int.1 Definitions

inc:int:def:sec In order to carry out Hilbert's project of formalizing mathematics and showing that such a formalization is consistent and complete, the first order of business would be that of picking a language, logical framework, and a system of axioms. For our purposes, let us suppose that mathematics can be formalized in a first-order language, i.e., that there is some set of constant symbols, function symbols, and predicate symbols which, together with the connectives and quatifiers of first-order logic, allow us to express the claims of mathematics. Most people agree that such a language exists: the language of set theory, in which $\in$ is the only non-logical symbol. That such a simple language is so expressive is of course a very implausible claim at first sight, and it took a lot of work to establish that practically of all mathematics can be expressed in this very austere vocabulary. To keep things simple, for now, let's restrict our discussion to arithmetic, so the part of mathematics that just deals with the natural numbers $\mathbb{N}$. The natural language in which to express facts of arithmetic is $\mathcal{L}_A$. $\mathcal{L}_A$ contains a single two-place predicate symbol $<$, a single constant symbol o, one one-place function symbol $\prime$, and two two-place function symbols $+$ and $\times$.

Definition int.1. A set of sentences Γ is a *theory* if it is closed under entailment, i.e., if $\Gamma = \{\varphi : \Gamma \vDash \varphi\}$.

There are two easy ways to specify theories. One is as the set of sentences true in some structure. For instance, consider the structure for $\mathcal{L}_A$ in which the domain is $\mathbb{N}$ and all non-logical symbols are interpreted as you would expect.

Definition int.2. The *standard model of arithmetic* is the structure $\mathfrak{N}$ defined as follows:

1. $|\mathfrak{N}| = \mathbb{N}$
2. $\mathsf{o}^{\mathfrak{N}} = 0$
3. $\prime^{\mathfrak{N}}(n) = n + 1$ for all $n \in \mathbb{N}$
4. $+^{\mathfrak{N}}(n, m) = n + m$ for all $n, m \in \mathbb{N}$
5. $\times^{\mathfrak{N}}(n, m) = n \cdot m$ for all $n, m \in \mathbb{N}$
6. $<^{\mathfrak{N}} = \{\langle n, m\rangle : n \in \mathbb{N}, m \in \mathbb{N}, n < m\}$

Note the difference between $\times$ and $\cdot$: $\times$ is a symbol in the language of arithmetic. Of course, we've chosen it to remind us of multiplication, but $\times$ is not the multiplication operation but a two-place function symbol (officially, f^2_1. By contrast, $\cdot$ *is* the ordinary multiplication function. When you see something like $n \cdot m$, we mean the product of the numbers n and m; when you see something like $x \times y$ we are talking about a term in the language of arithmetic. In the standard model, the function symbol times is interpreted as the function $\cdot$ on the natural numbers. For addition, we use $+$ as both the function symbol of the language of arithmetic, and the addition function on the natural numbers. Here you have to use the context to determine what is meant.

Definition int.3. The theory of *true arithmetic* is the set of sentences satisfied in the standard model of arithmetic, i.e.,

$$\mathbf{TA} = \{\varphi : \mathfrak{N} \vDash \varphi\}.$$

$\mathbf{TA}$ is a theory, for whenever $\mathbf{TA} \vDash \varphi$, φ is satisfied in every structure which satisfies $\mathbf{TA}$. Since $\mathfrak{M} \vDash \mathbf{TA}$, $\mathfrak{M} \vDash \varphi$, and so $\varphi \in \mathbf{TA}$.

The other way to specify a theory Γ is as the set of sentences entailed by some set of sentences Γ_0. In that case, Γ is the "closure" of Γ_0 under entailment. Specifying a theory this way is only interesting if Γ_0 is explicitly specified, e.g., if the elements of Γ_0 are listed. At the very least, Γ_0 has to be decidable, i.e., there has to be a computable test for when a sentence counts as an element of Γ_0 or not. We call the sentences in Γ_0 *axioms* for Γ, and Γ *axiomatized* by Γ_0.

Definition int.4. A theory Γ is *axiomatized* by Γ_0 iff

$$\Gamma = \{\varphi : \Gamma_0 \vDash \varphi\}$$

Definition int.5. The theory $\mathbf{Q}$ axiomatized by the following sentences is known as "Robinson's $\mathbf{Q}$" and is a very simple theory of arithmetic.

$$\forall x\, \forall y\, (x' = y' \rightarrow x = y) \tag{Q_1}$$
$$\forall x\, \mathsf{o} \neq x' \tag{Q_2}$$
$$\forall x\, (x = \mathsf{o} \lor \exists y\, x = y') \tag{Q_3}$$
$$\forall x\, (x + \mathsf{o}) = x \tag{Q_4}$$
$$\forall x\, \forall y\, (x + y') = (x + y)' \tag{Q_5}$$
$$\forall x\, (x \times \mathsf{o}) = \mathsf{o} \tag{Q_6}$$
$$\forall x\, \forall y\, (x \times y') = ((x \times y) + x) \tag{Q_7}$$
$$\forall x\, \forall y\, (x < y \leftrightarrow \exists z\, (z' + x) = y) \tag{Q_8}$$

The set of sentences $\{Q_1, \dots, Q_8\}$ are the axioms of $\mathbf{Q}$, so $\mathbf{Q}$ consists of all sentences entailed by them:

$$\mathbf{Q} = \{\varphi : \{Q_1, \dots, Q_8\} \vDash \varphi\}.$$

Definition int.6. Suppose $\varphi(x)$ is a formula in $\mathcal{L}_A$ with free variables x and y_1, ..., y_n. Then any sentence of the form

$$\forall y_1 \dots \forall y_n\, ((\varphi(\mathsf{o}) \land \forall x\, (\varphi(x) \rightarrow \varphi(x'))) \rightarrow \forall x\, \varphi(x))$$

is an instance of the *induction schema*.

Peano arithmetic $\mathbf{PA}$ is the theory axiomatized by the axioms of $\mathbf{Q}$ together with all instances of the induction schema.

Every instance of the induction schema is true in $\mathfrak{N}$. This is easiest to see if the formula φ only has one free variable x. Then $\varphi(x)$ defines a subset X_A of $\mathbb{N}$ in $\mathfrak{N}$. X_A is the set of all $n \in \mathbb{N}$ such that $\mathfrak{N}, s \vDash \varphi(x)$ when $s(x) = n$. The corresponding instance of the induction schema is explanation

$$((\varphi(\mathsf{o}) \land \forall x\,(\varphi(x) \to \varphi(x'))) \to \forall x\,\varphi(x)).$$

If its antecedent is true in $\mathfrak{N}$, then $0 \in X_A$ and, whenever $n \in X_A$, so is $n + 1$. Since $0 \in X_A$, we get $1 \in X_A$. With $1 \in X_A$ we get $2 \in X_A$. And so on. So for every $n \in \mathbb{N}$, $n \in X_A$. But this means that $\forall x\,\varphi(x)$ is satisfied in $\mathfrak{N}$.

Both **Q** and **PA** are axiomatized theories. The big question is, how strong are they? For instance, can **PA** prove all the truths about $\mathbb{N}$ that can be expressed in $\mathcal{L}_A$? Specifically, do the axioms of **PA** settle all the questions that can be formulated in $\mathcal{L}_A$?

Another way to put this is to ask: Is **PA** = **TA**? **TA** obviously does prove (i.e., it includes) all the truths about $\mathbb{N}$, and it settles all the questions that can be formulated in $\mathcal{L}_A$, since if φ is a sentence in $\mathcal{L}_A$, then either $\mathfrak{N} \vDash \varphi$ or $\mathfrak{N} \vDash \lnot\varphi$, and so either **TA** $\vDash \varphi$ or **TA** $\vDash \lnot\varphi$. Call such a theory *complete*.

Definition int.7. A theory Γ is *complete* iff for every sentence φ in its language, either $\Gamma \vDash \varphi$ or $\Gamma \vDash \lnot\varphi$.

By the Completeness Theorem, $\Gamma \vDash \varphi$ iff $\Gamma \vdash \varphi$, so Γ is complete iff for every sentence φ in its language, either $\Gamma \vdash \varphi$ or $\Gamma \vdash \lnot\varphi$. explanation

Another question we are led to ask is this: Is there a computational procedure we can use to test if a sentence is in **TA**, in **PA**, or even just in **Q**? We can make this more precise by defining when a set (e.g., a set of sentences) is decidable.

Definition int.8. A set X is *decidable* iff there is a computational procedure which on input x returns 1 if $x \in X$ and 0 otherwise.

So our question becomes: Is **TA** (**PA**, **Q**) decidable?

The answer to all these questions will be: no. None of these theories are decidable. However, this phenomenon is not specific to these particular theories. In fact, *any* theory that satisfies certain conditions is subject to the same results. One of these conditions, which **Q** and **PA** satisfy, is that they are axiomatized by a decidable set of axioms.

Definition int.9. A theory is *axiomatizable* if it is axiomatized by a decidable set of axioms.

Example int.10. Any theory axiomatized by a finite set of sentences is axiomatizable, since any finite set is decidable. Thus, **Q**, for instance, is axiomatizable.

Schematically axiomatized theories like **PA** are also axiomatizable. For to test if ψ is among the axioms of **PA**, i.e., to compute the function χ_X where $\chi_X(\psi) = 1$ if ψ is an axiom of **PA** and $= 0$ otherwise, we can do the following:

First, check if ψ is one of the axioms of **Q**. If it is, the answer is "yes" and the value of $\chi_X(\psi) = 1$. If not, test if it is an instance of the induction schema. This can be done systematically; in this case, perhaps it's easiest to see that it can be done as follows: Any instance of the induction schema begins with a number of universal quantifiers, and then a sub-formula that is a conditional. The consequent of that conditional is $\forall x\, \varphi(x, y_1, \dots, y_n)$ where x and y_1, ..., y_n are all the free variables of φ and the initial quantifiers of ψ bind the variables y_1, ..., y_n. Once we have extracted this φ and checked that its free variables match the variables bound by the universal qauntifiers at the front and $\forall x$, we go on to check that the antecedent of the conditional matches

$$\varphi(\mathsf{o}, y_1, \dots, y_n) \land \forall x\, (\varphi(x, y_1, \dots, y_n) \to \varphi(x', y_1, \dots, y_n))$$

Again, if it does, ψ is an instance of the induction schema, and if it doesn't, ψ isn't.

In answering this question—and the more general question of which theories are complete or decidable—it will be useful to consider also the following definition. Recall that a set X is enumerable iff it is empty or if there is a surjective function $f\colon \mathbb{N} \to X$. Such a function is called an enumeration of X.

Definition int.11. A set X is called *computably enumerable* (c.e. for short) iff it is empty or it has a computable enumeration.

In addition to axiomatizability, another condition on theories to which the incompleteness theorems apply will be that they are strong enough to prove basic facts about computable functions and decidable relations. By "basic facts," we mean sentences which express what the values of computable functions are for each of their arguments. And by "strong enough" we mean that the theories in question count these sentences among its theorems. For instance, consider a prototypical computable function: addition. The value of $+$ for arguments 2 and 3 is 5, i.e., $2 + 3 = 5$. A sentence in the language of arithmetic that expresses that the value of $+$ for arguments 2 and 3 is 5 is: $(\overline{2} + \overline{3}) = \overline{5}$. And, e.g., **Q** proves this sentence. More generally, we would like there to be, for each computable function $f(x_1, x_2)$ a formula $\varphi_f(x_1, x_2, y)$ in $\mathcal{L}_A$ such that $\mathbf{Q} \vdash \varphi_f(\overline{n_1}, \overline{n_2}, \overline{m})$ whenever $f(n_1, n_2) = m$. In this way, **Q** proves that the value of f for arguments n_1, n_2 is m. In fact, we require that it proves a bit more, namely that no other number is the value of f for arguments n_1, n_2. And the same goes for decidable relations. This is made precise in the following two definitions.

Definition int.12. A formula $\varphi(x_1, \dots, x_k, y)$ *represents* the function $f\colon \mathbb{N}^k \to \mathbb{N}$ in Γ iff whenever $f(n_1, \dots, n_k) = m$, then

1. $\Gamma \vdash \varphi(\overline{n_1}, \dots, \overline{n_k}, \overline{m})$, and
2. $\Gamma \vdash \forall y(\varphi(\overline{n_1}, \dots, \overline{n_k}, y) \to y = \overline{m})$.

Definition int.13. A formula $\varphi(x_1, \dots, x_k)$ *represents* the relation $R \subseteq \mathbb{N}^k$ iff,

1. whenever $R(n_1, \dots, n_k)$, $\Gamma \vdash \varphi(\overline{n_1}, \dots, \overline{n_k})$, and
2. whenever not $R(n_1, \dots, n_k)$, $\Gamma \vdash \neg\varphi(\overline{n_1}, \dots, \overline{n_k})$.

A theory is "strong enough" for the incompleteness theorems to apply if it represents all computable functions and all decidable relations. **Q** and its extensions satisfy this condition, but it will take us a while to establish this—it's a non-trivial fact about the kinds of things **Q** can prove, and it's hard to show because **Q** has only a few axioms from which we'll have to prove all these facts. However, **Q** is a very weak theory. So although it's hard to prove that **Q** represents all computable functions, most interesting theories are stronger than **Q**, i.e., prove more than **Q** does. And if **Q** proves something, any stronger theory does; since **Q** represents all computable functions, every stronger theory does. This means that many interesting theories meet this condition of the incompleteness theorems. So our hard work will pay off, since it shows that the incompletess theorems apply to a wide range of theories. Certainly, any theory aiming to formalize "all of mathematics" must prove everything that **Q** proves, since it should at the very least be able to capture the results of elementary computations. So any theory that is a candidate for a theory of "all of mathematics" will be one to which the incompleteness theorems apply.

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Bibliography