

## inp.1 Rosser's Theorem

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sec Can we modify Gödel's proof to get a stronger result, replacing “ $\omega$ -consistent” with simply “consistent”? The answer is “yes,” using a trick discovered by Rosser. Rosser's trick is to use a “modified” **derivability** predicate  $\text{RProv}_T(y)$  instead of  $\text{Prov}_T(y)$ .

inc:inp:ros:  
thm:rosser **Theorem inp.1.** *Let  $\mathbf{T}$  be any consistent, axiomatizable theory extending  $\mathbf{Q}$ . Then  $\mathbf{T}$  is not complete.*

*Proof.* Recall that  $\text{Prov}_T(y)$  is defined as  $\exists x \text{Prf}_T(x, y)$ , where  $\text{Prf}_T(x, y)$  represents the decidable relation which holds iff  $x$  is the Gödel number of a **derivation** of the **sentence** with Gödel number  $y$ . The relation that holds between  $x$  and  $y$  if  $x$  is the Gödel number of a *refutation* of the sentence with Gödel number  $y$  is also decidable. Let  $\text{not}(x)$  be the primitive recursive function which does the following: if  $x$  is the code of a formula  $\varphi$ ,  $\text{not}(x)$  is a code of  $\neg\varphi$ . Then  $\text{Ref}_T(x, y)$  holds iff  $\text{Prf}_T(x, \text{not}(y))$ . Let  $\text{Ref}_T(x, y)$  represent it. Then, if  $\mathbf{T} \vdash \neg\varphi$  and  $\delta$  is a corresponding **derivation**,  $\mathbf{Q} \vdash \text{Ref}_T(\ulcorner\delta\urcorner, \ulcorner\varphi\urcorner)$ . We define  $\text{RProv}_T(y)$  as

$$\exists x (\text{Prf}_T(x, y) \wedge \forall z (z < x \rightarrow \neg \text{Ref}_T(z, y))).$$

Roughly,  $\text{RProv}_T(y)$  says “there is a proof of  $y$  in  $\mathbf{T}$ , and there is no shorter refutation of  $y$ .” Assuming  $\mathbf{T}$  is consistent,  $\text{RProv}_T(y)$  is true of the same numbers as  $\text{Prov}_T(y)$ ; but from the point of view of *provability* in  $\mathbf{T}$  (and we now know that there is a difference between truth and provability!) the two have different properties. If  $\mathbf{T}$  is *inconsistent*, then the two do *not* hold of the same numbers! ( $\text{RProv}_T(y)$  is often read as “ $y$  is Rosser provable.” Since, as just discussed, Rosser provability is not some special kind of provability—in inconsistent theories, there are **sentences** that are provable but not Rosser provable—this may be confusing. To avoid the confusion, you could instead read it as “ $y$  is shmovable.”)

By the fixed-point lemma, there is a formula  $\rho_{\mathbf{T}}$  such that

$$\mathbf{Q} \vdash \rho_{\mathbf{T}} \leftrightarrow \neg \text{RProv}_T(\ulcorner\rho_{\mathbf{T}}\urcorner). \quad (1)$$

In contrast to the proof of ??, here we claim that if  $\mathbf{T}$  is consistent,  $\mathbf{T}$  doesn't **derive**  $\rho_{\mathbf{T}}$ , and  $\mathbf{T}$  also doesn't **derive**  $\neg\rho_{\mathbf{T}}$ . (In other words, we don't need the assumption of  $\omega$ -consistency.)

First, let's show that  $\mathbf{T} \not\vdash \rho_{\mathbf{T}}$ . Suppose it did, so there is a **derivation** of  $\rho_{\mathbf{T}}$  from  $T$ ; let  $n$  be its Gödel number. Then  $\mathbf{Q} \vdash \text{Prf}_T(\bar{n}, \ulcorner\rho_{\mathbf{T}}\urcorner)$ , since  $\text{Prf}_T$  represents  $\text{Prf}_T$  in  $\mathbf{Q}$ . Also, for each  $k < n$ ,  $k$  is not the Gödel number of a **derivation** of  $\neg\rho_{\mathbf{T}}$ , since  $\mathbf{T}$  is consistent. So for each  $k < n$ ,  $\mathbf{Q} \vdash \neg \text{Ref}_T(\bar{k}, \ulcorner\rho_{\mathbf{T}}\urcorner)$ . By ??,  $\mathbf{Q} \vdash \forall z (z < \bar{n} \rightarrow \neg \text{Ref}_T(z, \ulcorner\rho_{\mathbf{T}}\urcorner))$ . Thus,

$$\mathbf{Q} \vdash \exists x (\text{Prf}_T(x, \ulcorner\rho_{\mathbf{T}}\urcorner) \wedge \forall z (z < x \rightarrow \neg \text{Ref}_T(z, \ulcorner\rho_{\mathbf{T}}\urcorner))),$$

but that's just  $\text{RProv}_T(\ulcorner \rho_T \urcorner)$ . By eq. (1),  $\mathbf{Q} \vdash \neg \rho_T$ . Since  $\mathbf{T}$  extends  $\mathbf{Q}$ , also  $\mathbf{T} \vdash \neg \rho_T$ . We've assumed that  $\mathbf{T} \vdash \rho_T$ , so  $\mathbf{T}$  would be inconsistent, contrary to the assumption of the theorem.

Now, let's show that  $\mathbf{T} \not\vdash \neg \rho_T$ . Again, suppose it did, and suppose  $n$  is the Gödel number of a **derivation** of  $\neg \rho_T$ . Then  $\text{Ref}_T(n, \# \rho_T^\#)$  holds, and since  $\text{Ref}_T$  represents  $\text{Ref}_T$  in  $\mathbf{Q}$ ,  $\mathbf{Q} \vdash \text{Ref}_T(\bar{n}, \ulcorner \rho_T \urcorner)$ . We'll again show that  $\mathbf{T}$  would then be inconsistent because it would also **derive**  $\rho_T$ . Since

$$\mathbf{Q} \vdash \rho_T \leftrightarrow \neg \text{RProv}_T(\ulcorner \rho_T \urcorner),$$

and since  $\mathbf{T}$  extends  $\mathbf{Q}$ , it suffices to show that

$$\mathbf{Q} \vdash \neg \text{RProv}_T(\ulcorner \rho_T \urcorner).$$

The **sentence**  $\neg \text{RProv}_T(\ulcorner \rho_T \urcorner)$ , i.e.,

$$\neg \exists x (\text{Prf}_T(x, \ulcorner \rho_T \urcorner) \wedge \forall z (z < x \rightarrow \neg \text{Ref}_T(z, \ulcorner \rho_T \urcorner))),$$

is logically equivalent to

$$\forall x (\text{Prf}_T(x, \ulcorner \rho_T \urcorner) \rightarrow \exists z (z < x \wedge \text{Ref}_T(z, \ulcorner \rho_T \urcorner))).$$

We argue informally using logic, making use of facts about what  $\mathbf{Q}$  **derives**. Suppose  $x$  is arbitrary and  $\text{Prf}_T(x, \ulcorner \rho_T \urcorner)$ . We already know that  $\mathbf{T} \not\vdash \rho_T$ , and so for every  $k$ ,  $\mathbf{Q} \vdash \neg \text{Prf}_T(\bar{k}, \ulcorner \rho_T \urcorner)$ . Thus, for every  $k$  it follows that  $x \neq \bar{k}$ . In particular, we have (a) that  $x \neq \bar{n}$ . We also have  $\neg(x = \bar{0} \vee x = \bar{1} \vee \dots \vee x = \overline{n-1})$  and so by ??, (b)  $\neg(x < \bar{n})$ . By ??,  $\bar{n} < x$ . Since  $\mathbf{Q} \vdash \text{Ref}_T(\bar{n}, \ulcorner \rho_T \urcorner)$ , we have  $\bar{n} < x \wedge \text{Ref}_T(\bar{n}, \ulcorner \rho_T \urcorner)$ , and from that  $\exists z (z < x \wedge \text{Ref}_T(z, \ulcorner \rho_T \urcorner))$ . Since  $x$  was arbitrary we get, as required, that

$$\forall x (\text{Prf}_T(x, \ulcorner \rho_T \urcorner) \rightarrow \exists z (z < x \wedge \text{Ref}_T(z, \ulcorner \rho_T \urcorner))). \quad \square$$

**Problem inp.1.** Two sets  $A$  and  $B$  of natural numbers are said to be *computably inseparable* if there is no **decidable** set  $X$  such that  $A \subseteq X$  and  $B \subseteq \bar{X}$  ( $\bar{X}$  is the complement,  $\mathbb{N} \setminus X$ , of  $X$ ). Let  $\mathbf{T}$  be a consistent **axiomatizable** extension of  $\mathbf{Q}$ . Suppose  $A$  is the set of Gödel numbers of **sentences** provable in  $\mathbf{T}$  and  $B$  the set of Gödel numbers of sentences refutable in  $\mathbf{T}$ . Prove that  $A$  and  $B$  are computably inseparable.

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## Bibliography