inp.1 The Fixed-Point Lemma

inc:inp:fix: sec The fixed-point lemma says that for any formula $\psi(x)$, there is a sentence φ such that $\mathbf{T} \vdash \varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)$, provided $\mathbf{T}$ extends $\mathbf{Q}$. In the case of the liar sentence, we'd want φ to be equivalent (provably in $\mathbf{T}$) to "$\ulcorner\varphi\urcorner$ is false," i.e., the statement that ${}^{\#}\varphi^{\#}$ is the Gödel number of a false sentence. To understand the idea of the proof, it will be useful to compare it with Quine's informal gloss of φ as, "'yields a falsehood when preceded by its own quotation' yields a falsehood when preceded by its own quotation." The operation of taking an expression, and then forming a sentence by preceding this expression by its own quotation may be called *diagonalizing* the expression, and the result its diagonalization. So, the diagonalization of 'yields a falsehood when preceded by its own quotation' is "'yields a falsehood when preceded by its own quotation' yields a falsehood when preceded by its own quotation." Now note that Quine's liar sentence is not the diagonalization of 'yields a falsehood' but of 'yields a falsehood when preceded by its own quotation.' So the property being diagonalized to yield the liar sentence itself involves diagonalization! explanation

In the language of arithmetic, we form quotations of a formula with one free variable by computing its Gödel numbers and then substituting the standard numeral for that Gödel number into the free variable. The diagonalization of $\alpha(x)$ is $\alpha(\overline{n})$, where $n = {}^{\#}\alpha(x)^{\#}$. (From now on, let's abbreviate $\overline{{}^{\#}\alpha(x)^{\#}}$ as $\ulcorner\alpha(x)\urcorner$.) So if $\psi(x)$ is "is a falsehood," then "yields a falsehood if preceded by its own quotation," would be "yields a falsehood when applied to the Gödel number of its diagonalization." If we had a symbol diag for the function $\mathrm{diag}(n)$ which computes the Gödel number of the diagonalization of the formula with Gödel number n, we could write $\alpha(x)$ as $\psi(\mathit{diag}(x))$. And Quine's version of the liar sentence would then be the diagonalization of it, i.e., $\alpha(\ulcorner\alpha(x)\urcorner)$ or $\psi(\mathit{diag}(\ulcorner\psi(\mathit{diag}(x))\urcorner))$. Of course, $\psi(x)$ could now be any other property, and the same construction would work. For the incompleteness theorem, we'll take $\psi(x)$ to be "x is not derivable in $\mathbf{T}$." Then $\alpha(x)$ would be "yields a sentence not derivable in $\mathbf{T}$ when applied to the Gödel number of its diagonalization."

To formalize this in $\mathbf{T}$, we have to find a way to formalize diag. The function $\mathrm{diag}(n)$ is computable, in fact, it is primitive recursive: if n is the Gödel number of a formula $\alpha(x)$, $\mathrm{diag}(n)$ returns the Gödel number of $\alpha(\ulcorner\alpha(x)\urcorner)$. (Recall, $\ulcorner\alpha(x)\urcorner$ is the standard numeral of the Gödel number of $\alpha(x)$, i.e., $\overline{{}^{\#}\alpha(x)^{\#}}$). If diag were a function symbol in $\mathbf{T}$ representing the function diag, we could take φ to be the formula $\psi(\mathit{diag}(\ulcorner\psi(\mathit{diag}(x))\urcorner))$. Notice that

$$\begin{aligned}\mathrm{diag}({}^{\#}\psi(\mathit{diag}(x))^{\#}) &= {}^{\#}\psi(\mathit{diag}(\ulcorner\psi(\mathit{diag}(x))\urcorner))^{\#}\\ &= {}^{\#}\varphi^{\#}.\end{aligned}$$

Assuming $\mathbf{T}$ can derive

$$\mathit{diag}(\ulcorner\psi(\mathit{diag}(x))\urcorner) = \ulcorner\varphi\urcorner,$$

it can derive $\psi(\mathit{diag}(\ulcorner\psi(\mathit{diag}(x))\urcorner)) \leftrightarrow \psi(\ulcorner\varphi\urcorner)$. But the left hand side is, by definition, φ.

Of course, diag will in general not be a function symbol of $\mathbf{T}$, and certainly is not one of $\mathbf{Q}$. But, since diag is computable, it is *representable* in $\mathbf{Q}$ by some formula $\theta_{\text{diag}}(x, y)$. So instead of writing $\psi(\mathit{diag}(x))$ we can write $\exists y\,(\theta_{\text{diag}}(x, y) \land \psi(y))$. Otherwise, the proof sketched above goes through, and in fact, it goes through already in $\mathbf{Q}$.

Lemma inp.1. *Let $\psi(x)$ be any formula with one free variable x. Then there is a sentence φ such that $\mathbf{Q} \vdash \varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)$.* *inc:inp:fix: lem:fixed-point*

Proof. Given $\psi(x)$, let $\alpha(x)$ be the formula $\exists y\,(\theta_{\text{diag}}(x, y) \land \psi(y))$ and let φ be its diagonalization, i.e., the formula $\alpha(\ulcorner\alpha(x)\urcorner)$.

Since θ_{diag} represents diag, and $\text{diag}({}^{\#}\alpha(x)^{\#}) = {}^{\#}\varphi^{\#}$, $\mathbf{Q}$ can derive

$$\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, \ulcorner\varphi\urcorner) \tag{1}$$
inc:inp:fix: repdiag1

$$\forall y\,(\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, y) \to y = \ulcorner\varphi\urcorner). \tag{2}$$
inc:inp:fix: repdiag2

Now we show that $\mathbf{Q} \vdash \varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner)$. We argue informally, using just logic and facts derivable in $\mathbf{Q}$.

First, suppose φ, i.e., $\alpha(\ulcorner\alpha(x)\urcorner)$. Going back to the definition of $\alpha(x)$, we see that $\alpha(\ulcorner\alpha(x)\urcorner)$ just is

$$\exists y\,(\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, y) \land \psi(y)).$$

Consider such a y. Since $\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, y)$, by eq. (2), $y = \ulcorner\varphi\urcorner$. So, from $\psi(y)$ we have $\psi(\ulcorner\varphi\urcorner)$.

Now suppose $\psi(\ulcorner\varphi\urcorner)$. By eq. (1), we have

$$\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, \ulcorner\varphi\urcorner) \land \psi(\ulcorner\varphi\urcorner).$$

It follows that

$$\exists y\,(\theta_{\text{diag}}(\ulcorner\alpha(x)\urcorner, y) \land \psi(y)).$$

But that's just $\alpha(\ulcorner\alpha(x)\urcorner)$, i.e., φ. □

digression You should compare this to the proof of the fixed-point lemma in computability theory. The difference is that here we want to define a *statement* in terms of itself, whereas there we wanted to define a *function* in terms of itself; this difference aside, it is really the same idea.

Problem inp.1. A formula $\varphi(x)$ is a *truth definition* if $\mathbf{Q} \vdash \psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$ for all sentences ψ. Show that no formula is a truth definition by using the fixed-point lemma.

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Bibliography