art.1 Derivations in Natural Deduction

inc:art:pnd: sec In order to arithmetize derivations, we must represent derivations as numbers. Since derivations are trees of formulas where each inference carries one or two labels, a recursive representation is the most obvious approach: we represent a derivation as a tuple, the components of which are the number of immediate sub-derivations leading to the premises of the last inference, the representations of these sub-derivations, and the end-formula, the discharge label of the last inference, and a number indicating the type of the last inference. explanation

Definition art.1. If δ is a derivation in natural deduction, then $^{\#}\delta^{\#}$ is defined inductively as follows:

1. If δ consists only of the assumption φ, then $^{\#}\delta^{\#}$ is $\langle 0, {}^{\#}\varphi^{\#}, n\rangle$. The number n is 0 if it is an undischarged assumption, and the numerical label otherwise.

2. If δ ends in an inference with one, two, or three premises, then $^{\#}\delta^{\#}$ is

$$\langle 1, {}^{\#}\delta_1{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle,$$
$$\langle 2, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle, \text{ or}$$
$$\langle 3, {}^{\#}\delta_1{}^{\#}, {}^{\#}\delta_2{}^{\#}, {}^{\#}\delta_3{}^{\#}, {}^{\#}\varphi^{\#}, n, k\rangle,$$

respectively. Here δ_1, δ_2, δ_3 are the sub-derivations ending in the premise(s) of the last inference in δ, φ is the conclusion of the last inference in δ, n is the discharge label of the last inference (0 if the inference does not discharge any assumptions), and k is given by the following table according to which rule was used in the last inference.

Rule:	$\wedge$Intro	$\wedge$Elim	$\vee$Intro	$\vee$Elim
k:	1	2	3	4
Rule:	$\to$Intro	$\to$Elim	$\neg$Intro	$\neg$Elim
k:	5	6	7	8
Rule:	$\bot_I$	$\bot_C$	$\forall$Intro	$\forall$Elim
k:	9	10	11	12
Rule:	$\exists$Intro	$\exists$Elim	$=$Intro	$=$Elim
k:	13	14	15	16

Example art.2. Consider the very simple derivation

$$\cfrac{\cfrac{[\varphi \wedge \psi]^1}{\varphi}\,\wedge\text{Elim}}{(\varphi \wedge \psi) \to \varphi}\,{}^{1}\to\text{Intro}$$

The Gödel number of the assumption would be $d_0 = \langle 0, {}^{\#}\varphi \wedge \psi^{\#}, 1\rangle$. The Gödel number of the derivation ending in the conclusion of $\wedge$Elim would be $d_1 = \langle 1, d_0, {}^{\#}\varphi^{\#}, 0, 2\rangle$ (1 since $\wedge$Elim has one premise, the Gödel number of conclusion φ, 0 because no assumption is discharged, and 2 is the number coding $\wedge$Elim). The Gödel number of the entire derivation then is $\langle 1, d_1, {}^{\#}((\varphi \wedge \psi) \to \varphi)^{\#}, 1, 5\rangle$, i.e.,

$$\langle 1, \langle 1, \langle 0, {}^{\#}(\varphi \wedge \psi)^{\#}, 1\rangle, {}^{\#}\varphi^{\#}, 0, 2\rangle, {}^{\#}((\varphi \wedge \psi) \to \varphi)^{\#}, 1, 5\rangle.$$

explanation

Having settled on a representation of derivations, we must also show that we can manipulate Gödel numbers of such derivations primitive recursively, and express their essential properties and relations. Some operations are simple: e.g., given a Gödel number d of a derivation, $\mathrm{EndFmla}(d) = (d)_{(d)_0+1}$ gives us the Gödel number of its end-formula, $\mathrm{DischargeLabel}(d) = (d)_{(d)_0+2}$ gives us the discharge label and $\mathrm{LastRule}(d) = (d)_{(d)_0+3}$ the number indicating the type of the last inference. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "δ is a derivation of φ from Γ" is a primitive recursive relation of the Gödel numbers of δ and φ.

Proposition art.3. *The following relations are primitive recursive:*

1. *φ occurs as an assumption in δ with label n.*
2. *All assumptions in δ with label n are of the form φ (i.e., we can discharge the assumption φ using label n in δ).*

Proof. We have to show that the corresponding relations between Gödel numbers of formulas and Gödel numbers of derivations are primitive recursive.

1. We want to show that $\mathrm{Assum}(x, d, n)$, which holds if x is the Gödel number of an assumption of the derivation with Gödel number d labelled n, is primitive recursive. This is the case if the derivation with Gödel number $\langle 0, x, n\rangle$ is a sub-derivation of d. Note that the way we code derivations is a special case of the coding of trees introduced in **??**, so the primitive recursive function $\mathrm{SubtreeSeq}(d)$ gives a sequence of Gödel numbers of all sub-derivations of d (of length a most d). So we can define

$$\mathrm{Assum}(x, d, n) \Leftrightarrow (\exists i < d)\ (\mathrm{SubtreeSeq}(d))_i = \langle 0, x, n\rangle.$$

2. We want to show that $\mathrm{Discharge}(x, d, n)$, which holds if all assumptions with label n in the derivation with Gödel number d all are the formula with Gödel number x. But this relation holds iff $(\forall y < d)\ (\mathrm{Assum}(y, d, n) \to y = x)$. □

Proposition art.4. *The property* $\mathrm{Correct}(d)$ *which holds iff the last inference in the derivation δ with Gödel number d is correct, is primitive recursive.* inc:art:pnd:prop:followsby

Proof. Here we have to show that for each rule of inference R the relation $\mathrm{FollowsBy}_R(d)$ is primitive recursive, where $\mathrm{FollowsBy}_R(d)$ holds iff d is the Gödel number of derivation δ, and the end-formula of δ follows by a correct application of R from the immediate sub-derivations of δ.

A simple case is that of the $\land$Intro rule. If δ ends in a correct $\land$Intro inference, it looks like this:

$$\cfrac{\begin{array}{cc}\vdots\,\delta_1 & \vdots\,\delta_2 \\ \varphi & \psi\end{array}}{\varphi \land \psi}\;\land\mathrm{Intro}$$

Then the Gödel number d of δ is $\langle 2, d_1, d_2, {}^{\#}(\varphi \land \psi)^{\#}, 0, k\rangle$ where $\mathrm{EndFmla}(d_1) = {}^{\#}\varphi^{\#}$, $\mathrm{EndFmla}(d_2) = {}^{\#}B^{\#}$, $n = 0$, and $k = 1$. So we can define $\mathrm{FollowsBy}_{\land\mathrm{Intro}}(d)$ as

$$\begin{aligned}&(d)_0 = 2 \land \mathrm{DischargeLabel}(d) = 0 \land \mathrm{LastRule}(d) = 1 \land \\ &\mathrm{EndFmla}(d) = {}^{\#}({}^{\#} \frown \mathrm{EndFmla}((d)_1) \frown {}^{\#}\land^{\#} \frown \mathrm{EndFmla}((d)_2) \frown {}^{\#})^{\#}.\end{aligned}$$

Another simple example if the $=$Intro rule. Here the premise is an empty derivation, i.e., $(d)_1 = 0$, and no discharge label, i.e., $n = 0$. However, φ must be of the form $t = t$, for a closed term t. Here, a primitive recursive definition is

$$\begin{aligned}&(d)_0 = 1 \land (d)_1 = 0 \land \mathrm{DischargeLabel}(d) = 0 \land \\ &\qquad (\exists t < d)\ (\mathrm{ClTerm}(t) \land \mathrm{EndFmla}(d) = {}^{\#}{=}({}^{\#} \frown t \frown {}^{\#},^{\#} \frown t \frown {}^{\#})^{\#})\end{aligned}$$

For a more complicated example, $\mathrm{FollowsBy}_{\to\mathrm{Intro}}(d)$ holds iff the end-formula of δ is of the form $(\varphi \to \psi)$, where the end-formula of δ_1 is ψ, and any assumption in δ labelled n is of the form φ. We can express this primitive recursively by

$$\begin{aligned}&(d)_0 = 1 \land \\ &\qquad (\exists a < d)\ (\mathrm{Discharge}(a, (d)_1, \mathrm{DischargeLabel}(d)) \land \\ &\qquad \mathrm{EndFmla}(d) = ({}^{\#}({}^{\#} \frown a \frown {}^{\#}{\to}^{\#} \frown \mathrm{EndFmla}((d)_1) \frown {}^{\#})^{\#}))\end{aligned}$$

(Think of a as the Gödel number of φ).

For another example, consider $\exists$Intro. Here, the last inference in δ is correct iff there is a formula φ, a closed term t and a variable x such that $\varphi[t/x]$ is the end-formula of the derivation δ_1 and $\exists x\, \varphi$ is the conclusion of the last inference. So, $\mathrm{FollowsBy}_{\exists\mathrm{Intro}}(d)$ holds iff

$$\begin{aligned}&(d)_0 = 1 \land \mathrm{DischargeLabel}(d) = 0 \land \\ &\qquad (\exists a < d)\ (\exists x < d)\ (\exists t < d)\ (\mathrm{ClTerm}(t) \land \mathrm{Var}(x) \land \\ &\qquad \mathrm{Subst}(a, t, x) = \mathrm{EndFmla}((d)_1) \land \mathrm{EndFmla}(d) = ({}^{\#}\exists^{\#} \frown x \frown a)).\end{aligned}$$

We then define Correct(d) as

$$\begin{aligned}
&\mathrm{Sent}(\mathrm{EndFmla}(d)) \land \\
&\qquad (\mathrm{LastRule}(d) = 1 \land \mathrm{FollowsBy}_{\land\mathrm{Intro}}(d)) \lor \dots \lor \\
&\qquad (\mathrm{LastRule}(d) = 16 \land \mathrm{FollowsBy}_{=\mathrm{Elim}}(d)) \lor \\
&\qquad\qquad (\exists n < d)\, (\exists x < d)\, (d = \langle 0, x, n\rangle).
\end{aligned}$$

The first line ensures that the end-formula of d is a sentence. The last line covers the case where d is just an assumption. □

Problem art.1. Define the following properties as in Proposition art.4:

1. $\mathrm{FollowsBy}_{\to\mathrm{Elim}}(d)$,
2. $\mathrm{FollowsBy}_{=\mathrm{Elim}}(d)$,
3. $\mathrm{FollowsBy}_{\lor\mathrm{Elim}}(d)$,
4. $\mathrm{FollowsBy}_{\forall\mathrm{Intro}}(d)$.

For the last one, you will have to also show that you can test primitive recursively if the last inference of the derivation with Gödel number d satisfies the eigenvariable condition, i.e., the eigenvariable a of the ∀Intro inference occurs neither in the end-formula of d nor in an open assumption of d. You may use the primitive recursive predicate OpenAssum from Proposition art.6 for this.

Proposition art.5. *The relation* Deriv(d) *which holds if d is the Gödel number of a correct derivation δ, is primitive recursive.* inc:art:pnd: prop:deriv

Proof. A derivation δ is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-derivations ends in a correct inference. So, Deriv(d) iff

$$(\forall i < \mathrm{len}(\mathrm{SubtreeSeq}(d)))\ \mathrm{Correct}((\mathrm{SubtreeSeq}(d))_i)$$

□

Proposition art.6. *The relation* OpenAssum(z, d) *that holds if z is the Gödel number of an undischarged assumption φ of the derivation δ with Gödel number d, is primitive recursive.* inc:art:pnd: prop:openassum

Proof. An occurrence of an assumption is discharged if it occurs with label n in a sub-derivation of δ that ends in a rule with discharge label n. So φ is an undischarged assumption of δ if at least one of its occurrences is not discharged in δ. We must be careful: δ may contain both discharged and undischarged occurrences of φ.

Consider a sequence $\delta_0, \dots, \delta_k$ where $\delta_0 = d$, δ_k is the assumption $[\varphi]^n$ (for some n), and δ_i is an immediate sub-derivation of δ_{i+1}. If such a sequence exists in which no δ_i ends in an inference with discharge label n, then φ is an undischarged assumption of δ.

The primitive recursive function $\mathrm{SubtreeSeq}(d)$ provides us with a sequence of Gödel numbers of all sub-derivations of δ. Any sequence of Gödel numbers of sub-derivations of δ is a subsequence of it. Being a subsequence of is a primitive recursive relation: $\mathrm{Subseq}(s, s')$ holds iff $(\forall i < \mathrm{len}(s))\ \exists j < \mathrm{len}(s')\ (s)_i = (s)_j$. Being an immediate sub-derivation is as well: $\mathrm{Subderiv}(d, d')$ iff $(\exists j < (d')_0)\ d = (d')_j$. So we can define $\mathrm{OpenAssum}(z, d)$ by

$$\begin{aligned}(\exists s < \mathrm{SubtreeSeq}(d))\ (\mathrm{Subseq}(s, \mathrm{SubtreeSeq}(d)) \land (s)_0 = d \land \\ (\exists n < d)\ ((s)_{\mathrm{len}(s) \dot{-} 1} = \langle 0, z, n\rangle \land \\ (\forall i < (\mathrm{len}(s) \dot{-} 1))\ (\mathrm{Subderiv}((s)_i, (s)_{i+1})] \land \\ \mathrm{DischargeLabel}((s)_{i+1}) \neq n))). \quad \square\end{aligned}$$

Proposition art.7. *Suppose Γ is a primitive recursive set of sentences. Then the relation $\mathrm{Prf}_\Gamma(x, y)$ expressing "x is the code of a derivation δ of φ from undischarged assumptions in Γ and y is the Gödel number of φ" is primitive recursive.*

Proof. Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\mathrm{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of a natural deduction derivation with end formula φ and all undischarged assumptions in Γ is primitive recursive.

By Proposition art.5, the property $\mathrm{Deriv}(x)$ which holds iff x is the Gödel number of a correct derivation δ in natural deduction is primitive recursive. Thus we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned}\mathrm{Prf}_\Gamma(x, y) \Leftrightarrow \mathrm{Deriv}(x) \land \mathrm{EndFmla}(x) = y \land \\ (\forall z < x)\ (\mathrm{OpenAssum}(z, x) \to R_\Gamma(z)). \quad \square\end{aligned}$$

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Bibliography