art.1 Derivations in LK

inc:art:plk:sec In order to arithmetize derivations, we must represent derivations as numbers. explanation Since derivations are trees of sequents where each inference carries also a label, a recursive representation is the most obvious approach: we represent a derivation as a tuple, the components of which are the end-sequent, the label, and the representations of the sub-derivations leading to the premises of the last inference.

Definition art.1. If Γ is a finite sequence of sentences, $\Gamma = \langle \varphi_1, \dots, \varphi_n \rangle$, then ${}^{\#}\Gamma^{\#} = \langle {}^{\#}\varphi_1{}^{\#}, \dots, {}^{\#}\varphi_n{}^{\#} \rangle$.

If $\Gamma \Rightarrow \Delta$ is a sequent, then a Gödel number of $\Gamma \Rightarrow \Delta$ is

$$ {}^{\#}\Gamma \Rightarrow \Delta^{\#} = \langle {}^{\#}\Gamma^{\#}, {}^{\#}\Delta^{\#} \rangle $$

If π is a derivation in **LK**, then ${}^{\#}\pi^{\#}$ is defined as follows:

1. If π consists only of the initial sequent $\Gamma \Rightarrow \Delta$, then ${}^{\#}\pi^{\#}$ is
$$ \langle 0, {}^{\#}\Gamma \Rightarrow \Delta^{\#} \rangle. $$

2. If π ends in an inference with one or two premises, has $\Gamma \Rightarrow \Delta$ as its conclusion, and π_1 and π_2 are the immediate subproof ending in the premise of the last inference, then ${}^{\#}\pi^{\#}$ is
$$ \begin{aligned} &\langle 1, {}^{\#}\pi_1{}^{\#}, {}^{\#}\Gamma \Rightarrow \Delta^{\#}, k \rangle \text{ or} \\ &\langle 2, {}^{\#}\pi_1{}^{\#}, {}^{\#}\pi_2{}^{\#}, {}^{\#}\Gamma \Rightarrow \Delta^{\#}, k \rangle, \end{aligned} $$
respectively, where k is given by the following table according to which rule was used in the last inference:

Rule:	WL	WR	CL	CR	XL	XR
k:	1	2	3	4	5	6
Rule:	$\neg$L	$\neg$R	$\wedge$L	$\wedge$R	$\vee$L	$\vee$R
k:	7	8	9	10	11	12
Rule:	$\to$L	$\to$R	$\forall$L	$\forall$R	$\exists$L	$\exists$R
k:	13	14	15	16	17	18
Rule:	Cut	=				
k:	19	20				

Example art.2. Consider the very simple derivation

$$ \dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi}\,\wedge\text{L}}{\Rightarrow (\varphi \wedge \psi) \to \varphi}\,\to\text{R} $$

The Gödel number of the initial sequent would be $p_0 = \langle 0, {}^{\#}\varphi \Rightarrow \varphi^{\#} \rangle$. The Gödel number of the derivation ending in the conclusion of $\land$L would be $p_1 = \langle 1, p_0, {}^{\#}\varphi \land \psi \Rightarrow \varphi^{\#}, 9 \rangle$ (1 since $\land$L has one premise, the Gödel number of the conclusion $\varphi \land \psi \Rightarrow \varphi$, and 9 is the number coding $\land$L). The Gödel number of the entire derivation then is $\langle 1, p_1, {}^{\#}\!\Rightarrow (\varphi \land \psi) \to \varphi)^{\#}, 14 \rangle$, i.e.,

$$\langle 1, \langle 1, \langle 0, {}^{\#}\varphi \Rightarrow \varphi)^{\#} \rangle, {}^{\#}\varphi \land \psi \Rightarrow \varphi^{\#}, 9 \rangle, {}^{\#}\!\Rightarrow (\varphi \land \psi) \to \varphi^{\#}, 14 \rangle.$$

explanation

Having settled on a representation of derivations, we must also show that we can manipulate such derivations primitive recursively, and express their essential properties and relations so. Some operations are simple: e.g., given a Gödel number p of a derivation, $\mathrm{EndSeq}(p) = (p)_{(p)_0+1}$ gives us the Gödel number of its end-sequent and $\mathrm{LastRule}(p) = (p)_{(p)_0+2}$ the code of its last rule. The property $\mathrm{Sequent}(s)$ defined by

$$\mathrm{len}(s) = 2 \land (\forall i < \mathrm{len}((s)_0) + \mathrm{len}((s)_1))\ \mathrm{Sent}(((s)_0 \frown (s)_1)_i)$$

holds of s iff s is the Gödel number of a sequent consisting of sentences. Some are much harder. We'll at least sketch how to do this. The goal is to show that the relation "π is a derivation of φ from Γ" is a primitive recursive relation of the Gödel numbers of π and φ.

Proposition art.3. *The property* $\mathrm{Correct}(p)$ *which holds iff the last inference in the derivation* π *with Gödel number* p *is correct, is primitive recursive.* *inc:art:plk:prop:followsby*

Proof. $\Gamma \Rightarrow \Delta$ is an initial sequent if either there is a sentence φ such that $\Gamma \Rightarrow \Delta$ is $\varphi \Rightarrow \varphi$, or there is a term t such that $\Gamma \Rightarrow \Delta$ is $\emptyset \Rightarrow t = t$. In terms of Gödel numbers, $\mathrm{InitSeq}(s)$ holds iff

$$\begin{aligned}&(\exists x < s)\ (\mathrm{Sent}(x) \land s = \langle \langle x \rangle, \langle x \rangle \rangle) \lor \\ &(\exists t < s)\ (\mathrm{Term}(t) \land s = \langle 0, \langle {}^{\#}{=}({}^{\#} \frown t \frown {}^{\#},{}^{\#} \frown t \frown {}^{\#}){}^{\#} \rangle \rangle).\end{aligned}$$

We also have to show that for each rule of inference R the relation $\mathrm{FollowsBy}_R(p)$ is primitive recursive, where $\mathrm{FollowsBy}_R(p)$ holds iff p is the Gödel number of derivation π, and the end-sequent of π follows by a correct application of R from the immediate sub-derivations of π.

A simple case is that of the $\land$R rule. If π ends in a correct $\land$R inference, it looks like this:

$$\dfrac{\begin{matrix}\vdots\,\pi_1 \\ \Gamma \Rightarrow \Delta, \varphi\end{matrix} \qquad \begin{matrix}\vdots\,\pi_2 \\ \Gamma \Rightarrow \Delta, \psi\end{matrix}}{\Gamma \Rightarrow \Delta, \varphi \land \psi}\ \land\mathrm{R}$$

So, the last inference in the derivation π is a correct application of $\land$R iff there are sequences of sentences Γ and Δ as well as two sentences φ and ψ such that the end-sequent of π_1 is $\Gamma \Rightarrow \Delta, \varphi$, the end-sequent of π_2 is $\Gamma \Rightarrow \Delta, \psi$, and the end-sequent of π is $\Gamma \Rightarrow \Delta, \varphi \land \psi$. We just have to translate this into Gödel numbers. If $s = {}^{\#}\Gamma \Rightarrow \Delta^{\#}$ then $(s)_0 = {}^{\#}\Gamma^{\#}$ and $(s)_1 = {}^{\#}\Delta^{\#}$. So, $\mathrm{FollowsBy}_{\land\mathrm{R}}(p)$ holds iff

$$\begin{aligned}
&(\exists g < p)\ (\exists d < p)\ (\exists a < p)\ (\exists b < p) \\
&\quad \mathrm{EndSequent}(p) = \langle g, d \frown \langle {}^{\#}({}^{\#} \frown a \frown {}^{\#}\land^{\#} \frown b \frown {}^{\#}){}^{\#} \rangle\rangle \land \\
&\quad \mathrm{EndSequent}((p)_1) = \langle g, d \frown \langle a \rangle\rangle \land \\
&\quad \mathrm{EndSequent}((p)_2) = \langle g, d \frown \langle b \rangle\rangle \land \\
&\quad (p)_0 = 2 \land \mathrm{LastRule}(p) = 10.
\end{aligned}$$

The individual lines express, respectively, "there is a sequence (Γ) with Gödel number g, there is a sequence (Δ) with Gödel number d, a formula (φ) with Gödel number a, and a formula (ψ) with Gödel number b," such that "the end-sequent of π is $\Gamma \Rightarrow \Delta, \varphi \land \psi$," "the end-sequent of π_1 is $\Gamma \Rightarrow \Delta, \varphi$," "the end-sequent of π_2 is $\Gamma \Rightarrow \Delta, \psi$," and "$\pi$ has two immediate subderivations and the last inference rule is $\land$R (with number 10)."

The last inference in π is a correct application of $\exists$R iff there are sequences Γ and Δ, a formula φ, a variable x, and a term t, such that the end-sequent of π is $\Gamma \Rightarrow \Delta, \exists x\, \varphi$ and the end-sequent of π_1 is $\Gamma \Rightarrow \Delta, \varphi[t/x]$. So in terms of Gödel numbers, we have $\mathrm{FollowsBy}_{\exists\mathrm{R}}(p)$ iff

$$\begin{aligned}
&(\exists g < p)\ (\exists d < p)\ (\exists a < p)\ (\exists x < p)\ (\exists t < p) \\
&\quad \mathrm{EndSequent}(p) = \langle g, d \frown \langle {}^{\#}\exists^{\#} \frown x \frown a \rangle\rangle \land \\
&\quad \mathrm{EndSequent}((p)_1) = \langle g, d \frown \langle \mathrm{Subst}(a, t, x) \rangle\rangle \land \\
&\quad (p)_0 = 1 \land \mathrm{LastRule}(p) = 18.
\end{aligned}$$

We then define $\mathrm{Correct}(p)$ as

$$\begin{aligned}
&\mathrm{Sequent}(\mathrm{EndSequent}(p)) \land \\
&\qquad [(\mathrm{LastRule}(p) = 1 \land \mathrm{FollowsBy}_{\mathrm{WL}}(p)) \lor \cdots \lor \\
&\qquad\quad (\mathrm{LastRule}(p) = 20 \land \mathrm{FollowsBy}_{=}(p)) \lor \\
&\qquad\qquad (p)_0 = 0 \land \mathrm{InitialSeq}(\mathrm{EndSequent}(p))]
\end{aligned}$$

The first line ensures that the end-sequent of d is actually a sequent consisting of sentences. The last line covers the case where p is just an initial sequent. □

Problem art.1. Define the following properties as in Proposition art.3:

1. $\mathrm{FollowsBy}_{\mathrm{Cut}}(p)$,
2. $\mathrm{FollowsBy}_{\to\mathrm{L}}(p)$,
3. $\mathrm{FollowsBy}_{=}(p)$,
4. $\mathrm{FollowsBy}_{\forall\mathrm{R}}(p)$.

For the last one, you will have to also show that you can test primitive recursively if the last inference of the derivation with Gödel number p satisfies the eigenvariable condition, i.e., the eigenvariable a of the $\forall$R does not occur in the end-sequent.

Proposition art.4. *The relation* $\mathrm{Deriv}(p)$ *which holds if* p *is the Gödel number of a correct derivation* π*, is primitive recursive.* inc:art:plk:prop:deriv

Proof. A derivation π is correct if every one of its inferences is a correct application of a rule, i.e., if every one of its sub-derivations ends in a correct inference. So, $\mathrm{Deriv}(d)$ iff

$$(\forall i < \mathrm{len}(\mathrm{SubtreeSeq}(p)))\ \mathrm{Correct}((\mathrm{SubtreeSeq}(p))_i. \qquad \square$$

Proposition art.5. *Suppose* Γ *is a primitive recursive set of sentences. Then the relation* $\mathrm{Prf}_\Gamma(x, y)$ *expressing "*x *is the code of a derivation* π *of* $\Gamma_0 \Rightarrow \varphi$ *for some finite* $\Gamma_0 \subseteq \Gamma$ *and* y *is the Gödel number of* φ*" is primitive recursive.*

Proof. Suppose "$y \in \Gamma$" is given by the primitive recursive predicate $R_\Gamma(y)$. We have to show that $\mathrm{Prf}_\Gamma(x, y)$ which holds iff y is the Gödel number of a sentence φ and x is the code of an **LK**-derivation with end-sequent $\Gamma_0 \Rightarrow \varphi$ is primitive recursive.

By the previous proposition, the property $\mathrm{Deriv}(x)$ which holds iff x is the code of a correct derivation π in **LK** is primitive recursive. If x is such a code, then $\mathrm{EndSequent}(x)$ is the code of the end-sequent of π, and so $(\mathrm{EndSequent}(x))_0$ is the code of the left side of the end sequent and $(\mathrm{EndSequent}(x))_1$ the right side. So we can express "the right side of the end-sequent of π is φ" as $\mathrm{len}((\mathrm{EndSequent}(x))_1) = 1 \land ((\mathrm{EndSequent}(x))_1)_0 = x$. The left side of the end-sequent of π is of course automatically finite, we just have to express that every sentence in it is in Γ. Thus we can define $\mathrm{Prf}_\Gamma(x, y)$ by

$$\begin{aligned}\mathrm{Prf}_\Gamma(x, y) \Leftrightarrow\ & \mathrm{Deriv}(x) \land \\ & (\forall i < \mathrm{len}((\mathrm{EndSequent}(x))_0))\ R_\Gamma(((\mathrm{EndSequent}(x))_0)_i) \land \\ & \mathrm{len}((\mathrm{EndSequent}(x))_1) = 1 \land ((\mathrm{EndSequent}(x))_1)_0 = y. \qquad \square\end{aligned}$$

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Bibliography