

## tab.1 Soundness

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sec A **derivation** system, such as tableaux, is *sound* if it cannot **derive** things that explanation do not actually hold. Soundness is thus a kind of guaranteed safety property for **derivation** systems. Depending on which proof theoretic property is in question, we would like to know for instance, that

1. every **derivable**  $\varphi$  is valid;
2. if a **sentence** is **derivable** from some others, it is also a consequence of them;
3. if a set of **sentences** is inconsistent, it is unsatisfiable.

These are important properties of a **derivation** system. If any of them do not hold, the **derivation** system is deficient—it would **derive** too much. Consequently, establishing the soundness of a **derivation** system is of the utmost importance.

Because all these proof-theoretic properties are defined via closed **tableaux** of some kind or other, proving (1)–(3) above requires proving something about the semantic properties of closed **tableaux**. We will first define what it means for a **signed formula** to be satisfied in a structure, and then show that if a **tableau** is closed, no structure satisfies all its assumptions. (1)–(3) then follow as corollaries from this result.

**Definition tab.1.** A structure  $\mathfrak{M}$  satisfies a signed formula  $\mathbb{T}\varphi$  iff  $\mathfrak{M} \models \varphi$ , and it satisfies  $\mathbb{F}\varphi$  iff  $\mathfrak{M} \not\models \varphi$ .  $\mathfrak{M}$  satisfies a set of signed formulas  $\Gamma$  iff it satisfies every  $S\varphi \in \Gamma$ .  $\Gamma$  is *satisfiable* if there is a structure that satisfies it, and *unsatisfiable* otherwise.

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thm:tableau-soundness **Theorem tab.2 (Soundness).** *If  $\Gamma$  has a closed **tableau**,  $\Gamma$  is unsatisfiable.*

*Proof.* Let's call a branch of a **tableau** satisfiable iff the set of signed formulas on it is satisfiable, and let's call a **tableau** satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable **tableau** by one of the rules of inference always results in a satisfiable **tableau**. This will prove the theorem: any closed **tableau** results by applying rules of inference to the **tableau** consisting only of assumptions from  $\Gamma$ . So if  $\Gamma$  were satisfiable, any **tableau** for it would be satisfiable. A closed **tableau**, however, is clearly not satisfiable: every branch contains both  $\mathbb{T}\varphi$  and  $\mathbb{F}\varphi$ , and no structure can both satisfy and not satisfy  $\varphi$ .

Suppose we have a satisfiable **tableau**, i.e., a **tableau** with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the **tableau** has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended **tableau**, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let  $\Gamma$  be the set of **signed formulas** on that branch, and let  $S\varphi \in \Gamma$  be the **signed formula** to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e.,  $\Gamma$  together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

First, we consider the possible inferences with only one premise.

1. The branch is expanded by applying  $\neg\mathbb{T}$  to  $\mathbb{T}\neg\psi \in \Gamma$ . Then the extended branch contains the **signed formulas**  $\Gamma \cup \{\mathbb{F}\psi\}$ . Suppose  $\mathfrak{M} \models \Gamma$ . In particular,  $\mathfrak{M} \models \neg\psi$ . Thus,  $\mathfrak{M} \not\models \psi$ , i.e.,  $\mathfrak{M}$  satisfies  $\mathbb{F}\psi$ .
2. The branch is expanded by applying  $\neg\mathbb{F}$  to  $\mathbb{F}\neg\psi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\wedge\mathbb{T}$  to  $\mathbb{T}\psi \wedge \chi \in \Gamma$ , which results in two new **signed formulas** on the branch:  $\mathbb{T}\psi$  and  $\mathbb{T}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \models \psi \wedge \chi$ . Then  $\mathfrak{M} \models \psi$  and  $\mathfrak{M} \models \chi$ . This means that  $\mathfrak{M}$  satisfies both  $\mathbb{T}\psi$  and  $\mathbb{T}\chi$ .
4. The branch is expanded by applying  $\vee\mathbb{F}$  to  $\mathbb{T}\psi \vee \chi \in \Gamma$ : Exercise.
5. The branch is expanded by applying  $\rightarrow\mathbb{F}$  to  $\mathbb{T}\psi \rightarrow \chi \in \Gamma$ : This results in two new **signed formulas** on the branch:  $\mathbb{T}\psi$  and  $\mathbb{F}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \psi \rightarrow \chi$ . Then  $\mathfrak{M} \models \psi$  and  $\mathfrak{M} \not\models \chi$ . This means that  $\mathfrak{M}$  satisfies both  $\mathbb{T}\psi$  and  $\mathbb{F}\chi$ .
6. The branch is expanded by applying  $\forall\mathbb{T}$  to  $\mathbb{T}\forall x\psi(x) \in \Gamma$ : This results in a new **signed formula**  $\mathbb{T}\varphi(t)$  on the branch. Suppose  $\mathfrak{M} \models \Gamma$ , in particular,  $\mathfrak{M} \models \forall x\psi(x)$ . By ??,  $\mathfrak{M} \models \varphi(t)$ . Consequently,  $\mathfrak{M}$  satisfies  $\mathbb{T}\varphi(t)$ .
7. The branch is expanded by applying  $\forall\mathbb{F}$  to  $\mathbb{F}\forall x\psi(x) \in \Gamma$ : This results in a new **signed formula**  $\mathbb{F}\varphi(a)$  where  $a$  is a **constant symbol** not occurring in  $\Gamma$ . Since  $\Gamma$  is satisfiable, there is a  $\mathfrak{M}$  such that  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \forall x\psi(x)$ . We have to show that  $\Gamma \cup \{\mathbb{F}\varphi(a)\}$  is satisfiable. To do this, we define a suitable  $\mathfrak{M}'$  as follows.  
 By ??,  $\mathfrak{M} \not\models \forall x\psi(x)$  iff for some  $s$ ,  $\mathfrak{M}, s \not\models \psi(x)$ . Now let  $\mathfrak{M}'$  be just like  $\mathfrak{M}$ , except  $a^{\mathfrak{M}'} = s(x)$ . By ??, for any  $\mathbb{T}\chi \in \Gamma$ ,  $\mathfrak{M}' \models \chi$ , and for any  $\mathbb{F}\chi \in \Gamma$ ,  $\mathfrak{M}' \not\models \chi$ , since  $a$  does not occur in  $\Gamma$ .  
 By ??,  $\mathfrak{M}', s \not\models \varphi(x)$ . By ??,  $\mathfrak{M}', s \not\models \varphi(a)$ . Since  $\varphi(a)$  is a **sentence**, by ??,  $\mathfrak{M}' \not\models \varphi(a)$ , i.e.,  $\mathfrak{M}'$  satisfies  $\mathbb{F}\varphi(a)$ .
8. The branch is expanded by applying  $\exists\mathbb{T}$  to  $\mathbb{T}\exists x\psi(x) \in \Gamma$ : Exercise.
9. The branch is expanded by applying  $\exists\mathbb{F}$  to  $\mathbb{F}\exists x\psi(x) \in \Gamma$ : Exercise.

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying  $\wedge\mathbb{F}$  to  $\mathbb{F}\psi \wedge \chi \in \Gamma$ , which results in two branches, a left one continuing through  $\mathbb{F}\psi$  and a right one through  $\mathbb{F}\chi$ . Suppose  $\mathfrak{M} \models \Gamma$ , in particular  $\mathfrak{M} \not\models \psi \wedge \chi$ . Then  $\mathfrak{M} \not\models \psi$  or  $\mathfrak{M} \not\models \chi$ .

In the former case,  $\mathfrak{M}$  satisfies  $\mathbb{F}\psi$ , i.e.,  $\mathfrak{M}$  satisfies the formulas on the left branch. In the latter,  $\mathfrak{M}$  satisfies  $\mathbb{F}\chi$ , i.e.,  $\mathfrak{M}$  satisfies the formulas on the right branch.

2. The branch is expanded by applying  $\vee\mathbb{T}$  to  $\mathbb{T}\psi \vee \chi \in \Gamma$ : Exercise.
3. The branch is expanded by applying  $\rightarrow\mathbb{T}$  to  $\mathbb{T}\psi \rightarrow \chi \in \Gamma$ : Exercise.
4. The branch is expanded by Cut: This results in two branches, one containing  $\mathbb{T}\psi$ , the other containing  $\mathbb{F}\psi$ . Since  $\mathfrak{M} \models \Gamma$  and either  $\mathfrak{M} \models \psi$  or  $\mathfrak{M} \not\models \psi$ ,  $\mathfrak{M}$  satisfies either the left or the right branch.  $\square$

**Problem tab.1.** Complete the proof of **Theorem tab.2**.

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*cor:weak-soundness*

**Corollary tab.3.** *If  $\vdash \varphi$  then  $\varphi$  is valid.*

*fol:tab:sou:*  
*cor:entailment-soundness*

**Corollary tab.4.** *If  $\Gamma \vdash \varphi$  then  $\Gamma \models \varphi$ .*

*Proof.* If  $\Gamma \vdash \varphi$  then for some  $\psi_1, \dots, \psi_n \in \Gamma$ ,  $\{\mathbb{F}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\}$  has a closed **tableau**. By **Theorem tab.2**, every **structure**  $\mathfrak{M}$  either makes some  $\psi_i$  false or makes  $\varphi$  true. Hence, if  $\mathfrak{M} \models \Gamma$  then also  $\mathfrak{M} \models \varphi$ .  $\square$

*fol:tab:sou:*  
*cor:consistency-soundness*

**Corollary tab.5.** *If  $\Gamma$  is satisfiable, then it is consistent.*

*Proof.* We prove the contrapositive. Suppose that  $\Gamma$  is not consistent. Then there are  $\psi_1, \dots, \psi_n \in \Gamma$  and a closed **tableau** for  $\{\mathbb{T}\psi, \dots, \mathbb{T}\psi\}$ . By **Theorem tab.2**, there is no  $\mathfrak{M}$  such that  $\mathfrak{M} \models \psi_i$  for all  $i = 1, \dots, n$ . But then  $\Gamma$  is not satisfiable.  $\square$

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## Bibliography