Proposition tab.1. Tableaux with rules for identity are sound: no closed tableau is satisfiable.

Proof. We just have to show as before that if a tableau has a satisfiable branch, the branch resulting from applying one of the rules for = to it is also satisfiable. Let $\Gamma$ be the set of signed formulas on the branch, and let $\mathcal{M}$ be a structure satisfying $\Gamma$.

Suppose the branch is expanded using $=$, i.e., by adding the signed formula $\top t = t$. Trivially, $\mathcal{M} \models t = t$, so $\mathcal{M}$ also satisfies $\Gamma \cup \{ \top t = t \}$.

If the branch is expanded using $=\top$, we add a signed formula $S \varphi(t_2)$, but $\Gamma$ contains both $\top t_1 = t_2$ and $\top \varphi(t_1)$. Thus we have $\mathcal{M} \models t_1 = t_2$ and $\mathcal{M} \models \varphi(t_1)$. Let $s$ be a variable assignment with $s(x) = \text{Val}_{\mathcal{M}}(t_1)$. By $\text{??}, \mathcal{M}, s \models \varphi(t_1)$. Since $s \sim_x s$, by $\text{??}, \mathcal{M}, s \models \varphi(x)$. since $\mathcal{M} \models t_1 = t_2$, we have $\text{Val}_{\mathcal{M}}(t_1) = \text{Val}_{\mathcal{M}}(t_2)$, and hence $s(x) = \text{Val}_{\mathcal{M}}(t_2)$. By applying $\text{??}$ again, we also have $\mathcal{M}, s \models \varphi(t_2)$. By $\text{??}, \mathcal{M} \models \varphi(t_2)$. The case of $=\bot$ is treated similarly. □