tab.1 Examples of Tableaux

fol:tab:pro:sec

Example tab.1. Let's find a closed tableau for the sentence $(\varphi \land \psi) \to \varphi$.

We begin by writing the corresponding assumption at the top of the tableau.

$$1. \quad \mathbb{F}\,(\varphi \land \psi) \to \varphi \qquad \text{Assumption}$$

There is only one assumption, so only one signed formula to which we can apply a rule. (For every signed formula, there is always at most one rule that can be applied: it's the rule for the corresponding sign and main operator of the sentence.) In this case, this means, we must apply $\to\mathbb{F}$.

1.	$\mathbb{F}\,(\varphi \land \psi) \to \varphi$ ✓	Assumption
2.	$\mathbb{T}\,\varphi \land \psi$	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\varphi$	$\to\mathbb{F}\,1$

To keep track of which signed formulas we have applied their corresponding rules to, we write a checkmark next to the sentence. However, *only* write a checkmark if the rule has been applied to all open branches. Once a signed formula has had the corresponding rule applied in every open branch, we will not have to return to it and apply the rule again. In this case, there is only one branch, so the rule only has to be applied once. (Note that checkmarks are only a convenience for constructing tableaux and are not officially part of the syntax of tableaux.)

There is one new signed formula to which we can apply a rule: the $\mathbb{T}\,\varphi \land \psi$ on line 2. Applying the $\land\mathbb{T}$ rule results in:

1.	$\mathbb{F}\,(\varphi \land \psi) \to \varphi$ ✓	Assumption
2.	$\mathbb{T}\,\varphi \land \psi$ ✓	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\varphi$	$\land\mathbb{T}\,2$
5.	$\mathbb{T}\,\psi$	$\land\mathbb{T}\,2$
	$\otimes$	

Since the branch now contains both $\mathbb{T}\,\varphi$ (on line 4) and $\mathbb{F}\,\varphi$ (on line 3), the branch is closed. Since it is the only branch, the tableau is closed. We have found a closed tableau for $(\varphi \land \psi) \to \varphi$.

Example tab.2. Now let's find a closed tableau for $(\neg\varphi \lor \psi) \to (\varphi \to \psi)$.

We begin with the corresponding assumption:

$$1. \quad \mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi) \qquad \text{Assumption}$$

The one signed formula in this tableau has main operator $\to$ and sign $\mathbb{F}$, so we apply the $\to\mathbb{F}$ rule to it to obtain:

$$
\begin{array}{rcl}
1. & \mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)\ \checkmark & \text{Assumption} \\
2. & \mathbb{T}\,\neg\varphi \lor \psi & {\to}\mathbb{F}\,1 \\
3. & \mathbb{F}\,(\varphi \to \psi) & {\to}\mathbb{F}\,1
\end{array}
$$

We now have a choice as to whether to apply $\lor\mathbb{T}$ to line 2 or $\to\mathbb{F}$ to line 3. It actually doesn't matter which order we pick, as long as each signed formula has its corresponding rule applied in every branch. So let's pick the first one. The $\lor\mathbb{T}$ rule allows the tableau to branch, and the two conclusions of the rule will be the new signed formulas added to the two new branches. This results in:

$$
\begin{array}{rccl}
1. & \multicolumn{2}{c}{\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\neg\varphi \lor \psi\ \checkmark} & {\to}\mathbb{F}\,1 \\
3. & \multicolumn{2}{c}{\mathbb{F}\,(\varphi \to \psi)} & {\to}\mathbb{F}\,1 \\
 & \swarrow & \searrow & \\
4. & \mathbb{T}\,\neg\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,2
\end{array}
$$

We have not applied the $\to\mathbb{F}$ rule to line 3 yet: let's do that now. To save time, we apply it to both branches. Recall that we write a checkmark next to a signed formula only if we have applied the corresponding rule in every open branch. So it's a good idea to apply a rule at the end of every branch that contains the signed formula the rule applies to. That way we won't have to return to that signed formula lower down in the various branches.

$$
\begin{array}{rccl}
1. & \multicolumn{2}{c}{\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\neg\varphi \lor \psi\ \checkmark} & {\to}\mathbb{F}\,1 \\
3. & \multicolumn{2}{c}{\mathbb{F}\,(\varphi \to \psi)\ \checkmark} & {\to}\mathbb{F}\,1 \\
 & \swarrow & \searrow & \\
4. & \mathbb{T}\,\neg\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,2 \\
5. & \mathbb{T}\,\varphi & \mathbb{T}\,\varphi & {\to}\mathbb{F}\,3 \\
6. & \mathbb{F}\,\psi & \mathbb{F}\,\psi & {\to}\mathbb{F}\,3 \\
 & & \otimes &
\end{array}
$$

The right branch is now closed. On the left branch, we can still apply the $\neg\mathbb{T}$ rule to line 4. This results in $\mathbb{F}\,\varphi$ and closes the left branch:

$$
\begin{array}{rccl}
1. & \multicolumn{2}{c}{\mathbb{F}\,(\neg\varphi \lor \psi) \to (\varphi \to \psi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\neg\varphi \lor \psi\ \checkmark} & {\to}\mathbb{F}\,1 \\
3. & \multicolumn{2}{c}{\mathbb{F}\,(\varphi \to \psi)\ \checkmark} & {\to}\mathbb{F}\,1 \\
 & \swarrow & \searrow & \\
4. & \mathbb{T}\,\neg\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,2 \\
5. & \mathbb{T}\,\varphi & \mathbb{T}\,\varphi & {\to}\mathbb{F}\,3 \\
6. & \mathbb{F}\,\psi & \mathbb{F}\,\psi & {\to}\mathbb{F}\,3 \\
7. & \mathbb{F}\,\varphi & \otimes & \neg\mathbb{T}\,4 \\
 & \otimes & &
\end{array}
$$

Example tab.3. We can give tableaux for any number of signed formulas as assumptions. Often it is also necessary to apply more than one rule that allows branching; and in general a tableau can have any number of branches. For instance, consider a tableau for $\{\mathbb{T}\,\varphi \lor (\psi \land \chi), \mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)\}$. We start by applying the $\lor\mathbb{T}$ to the first assumption:

$$
\begin{array}{lccl}
1. & \multicolumn{2}{c}{\mathbb{T}\,\varphi \lor (\psi \land \chi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)} & \text{Assumption} \\
3. & \mathbb{T}\,\varphi & \mathbb{T}\,\psi \land \chi & \lor\mathbb{T}\,1
\end{array}
$$

Now we can apply the $\land\mathbb{F}$ rule to line 2. We do this on both branches simultaneously, and can therefore check off line 2:

$$
\begin{array}{lccccl}
1. & \multicolumn{4}{c}{\mathbb{T}\,\varphi \lor (\psi \land \chi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{4}{c}{\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)\ \checkmark} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\varphi} & \multicolumn{2}{c}{\mathbb{T}\,\psi \land \chi} & \lor\mathbb{T}\,1 \\
4. & \mathbb{F}\,\varphi \lor \psi & \mathbb{F}\,\varphi \lor \chi & \mathbb{F}\,\varphi \lor \psi & \mathbb{F}\,\varphi \lor \chi & \land\mathbb{F}\,2
\end{array}
$$

Now we can apply $\lor\mathbb{F}$ to all the branches containing $\varphi \lor \psi$:

$$
\begin{array}{lccccl}
1. & \multicolumn{4}{c}{\mathbb{T}\,\varphi \lor (\psi \land \chi)\ \checkmark} & \text{Assumption} \\
2. & \multicolumn{4}{c}{\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)\ \checkmark} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\varphi} & \multicolumn{2}{c}{\mathbb{T}\,\psi \land \chi} & \lor\mathbb{T}\,1 \\
4. & \mathbb{F}\,\varphi \lor \psi\ \checkmark & \mathbb{F}\,\varphi \lor \chi & \mathbb{F}\,\varphi \lor \psi\ \checkmark & \mathbb{F}\,\varphi \lor \chi & \land\mathbb{F}\,2 \\
5. & \mathbb{F}\,\varphi & & \mathbb{F}\,\varphi & & \lor\mathbb{F}\,4 \\
6. & \mathbb{F}\,\psi & & \mathbb{F}\,\psi & & \lor\mathbb{F}\,4 \\
 & \otimes & & & &
\end{array}
$$

The leftmost branch is now closed. Let's now apply $\lor\mathbb{F}$ to $\varphi \lor \chi$:

1.		$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓			Assumption
2.		$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓			Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\land\mathbb{F}\,2$
5.	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$		$\lor\mathbb{F}\,4$
6.	$\mathbb{F}\,\psi$		$\mathbb{F}\,\psi$		$\lor\mathbb{F}\,4$
7.	$\otimes$	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$	$\lor\mathbb{F}\,4$
8.		$\mathbb{F}\,\chi$		$\mathbb{F}\,\chi$	$\lor\mathbb{F}\,4$
		$\otimes$			

Note that we moved the result of applying $\lor\mathbb{F}$ a second time below for clarity. In this instance it would not have been needed, since the justifications would have been the same.

Two branches remain open, and $\mathbb{T}\,\psi \land \chi$ on line 3 remains unchecked. We apply $\land\mathbb{T}$ to it to obtain a closed tableau:

1.		$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓			Assumption
2.		$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓			Assumption
3.	$\mathbb{T}\,\varphi$		$\mathbb{T}\,\psi \land \chi$ ✓		$\lor\mathbb{T}\,1$
4.	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\mathbb{F}\,\varphi \lor \psi$ ✓	$\mathbb{F}\,\varphi \lor \chi$ ✓	$\land\mathbb{F}\,2$
5.	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\mathbb{F}\,\varphi$	$\lor\mathbb{F}\,4$
6.	$\mathbb{F}\,\psi$	$\mathbb{F}\,\chi$	$\mathbb{F}\,\psi$	$\mathbb{F}\,\chi$	$\lor\mathbb{F}\,4$
7.	$\otimes$	$\otimes$	$\mathbb{T}\,\psi$	$\mathbb{T}\,\psi$	$\land\mathbb{T}\,3$
8.			$\mathbb{T}\,\chi$	$\mathbb{T}\,\chi$	$\land\mathbb{T}\,3$
			$\otimes$	$\otimes$	

For comparison, here's a closed tableau for the same set of assumptions in which the rules are applied in a different order:

1.	$\mathbb{T}\,\varphi \lor (\psi \land \chi)$ ✓				Assumption
2.	$\mathbb{F}\,(\varphi \lor \psi) \land (\varphi \lor \chi)$ ✓				Assumption
3.	$\mathbb{F}\,\varphi \lor \psi$ ✓		$\mathbb{F}\,\varphi \lor \chi$ ✓		$\land\mathbb{F}\,2$
4.	$\mathbb{F}\,\varphi$		$\mathbb{F}\,\varphi$		$\lor\mathbb{F}\,3$
5.	$\mathbb{F}\,\psi$		$\mathbb{F}\,\chi$		$\lor\mathbb{F}\,3$
6.	$\mathbb{T}\,\varphi$	$\mathbb{T}\,\psi \land \chi$ ✓	$\mathbb{T}\,\varphi$	$\mathbb{T}\,\psi \land \chi$ ✓	$\lor\mathbb{T}\,1$
7.	$\otimes$	$\mathbb{T}\,\psi$	$\otimes$	$\mathbb{T}\,\psi$	$\land\mathbb{T}\,6$
8.		$\mathbb{T}\,\chi$		$\mathbb{T}\,\chi$	$\land\mathbb{T}\,6$
		$\otimes$		$\otimes$	

Problem tab.1. Give closed tableaux of the following:

1. $\mathbb{T}\,\varphi \land (\psi \land \chi), \mathbb{F}\,(\varphi \land \psi) \land \chi$.
2. $\mathbb{T}\,\varphi \lor (\psi \lor \chi), \mathbb{F}\,(\varphi \lor \psi) \lor \chi$.
3. $\mathbb{T}\,\varphi \to (\psi \to \chi), \mathbb{F}\,\psi \to (\varphi \to \chi)$.
4. $\mathbb{T}\,\varphi, \mathbb{F}\,\neg\neg\varphi$.

Problem tab.2. Give closed tableaux of the following:

1. $\mathbb{T}\,(\varphi \lor \psi) \to \chi, \mathbb{F}\,\varphi \to \chi$.
2. $\mathbb{T}\,(\varphi \to \chi) \land (\psi \to \chi), \mathbb{F}\,(\varphi \lor \psi) \to \chi$.
3. $\mathbb{F}\,\neg(\varphi \land \neg\varphi)$.
4. $\mathbb{T}\,\psi \to \varphi, \mathbb{F}\,\neg\varphi \to \neg\psi$.
5. $\mathbb{F}\,(\varphi \to \neg\varphi) \to \neg\varphi$.
6. $\mathbb{F}\,\neg(\varphi \to \psi) \to \neg\psi$.
7. $\mathbb{T}\,\varphi \to \chi, \mathbb{F}\,\neg(\varphi \land \neg\chi)$.
8. $\mathbb{T}\,\varphi \land \neg\chi, \mathbb{F}\,\neg(\varphi \to \chi)$.
9. $\mathbb{T}\,\varphi \lor \psi, \neg\psi, \mathbb{F}\,\varphi$.
10. $\mathbb{T}\,\neg\varphi \lor \neg\psi, \mathbb{F}\,\neg(\varphi \land \psi)$.
11. $\mathbb{F}\,(\neg\varphi \land \neg\psi) \to \neg(\varphi \lor \psi)$.
12. $\mathbb{F}\,\neg(\varphi \lor \psi) \to (\neg\varphi \land \neg\psi)$.

Problem tab.3. Give closed tableaux of the following:

1. $\mathbb{T}\,\neg(\varphi \to \psi), \mathbb{F}\,\varphi$.

2. $\mathbb{T}\,\neg(\varphi \land \psi), \mathbb{F}\,\neg\varphi \lor \neg\psi$.

3. $\mathbb{T}\,\varphi \to \psi, \mathbb{F}\,\neg\varphi \lor \psi$.

4. $\mathbb{F}\,\neg\neg\varphi \to \varphi$.

5. $\mathbb{T}\,\varphi \to \psi, \mathbb{T}\,\neg\varphi \to \psi, \mathbb{F}\,\psi$.

6. $\mathbb{T}\,(\varphi \land \psi) \to \chi, \mathbb{F}\,(\varphi \to \chi) \lor (\psi \to \chi)$.

7. $\mathbb{T}\,(\varphi \to \psi) \to \varphi, \mathbb{F}\,\varphi$.

8. $\mathbb{F}\,(\varphi \to \psi) \lor (\psi \to \chi)$.

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Bibliography