tab.1 Tableaux with Quantifiers

fol:tab:prq:sec

Example tab.1. When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be higher up in the finished tableau).

Let's see how we'd give a tableau for the sentence $\exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$. Starting as usual, we start by recording the assumption,

1.	$\mathbb{F}\, \exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$	Assumption

Since the main operator is $\to$, we apply the $\to\mathbb{F}$:

1.	$\mathbb{F}\, \exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\, \exists x\, \neg\varphi(x)$	$\to\mathbb{F}\, 1$
3.	$\mathbb{F}\, \neg\forall x\, \varphi(x)$	$\to\mathbb{F}\, 1$

The next line to deal with is 2. We use $\exists\mathbb{T}$. This requires a new constant symbol; since no constant symbols yet occur, we can pick any one, say, a.

1.	$\mathbb{F}\, \exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\, \exists x\, \neg\varphi(x)$ ✓	$\to\mathbb{F}\, 1$
3.	$\mathbb{F}\, \neg\forall x\, \varphi(x)$	$\to\mathbb{F}\, 1$
4.	$\mathbb{T}\, \neg\varphi(a)$	$\exists\mathbb{T}\, 2$

Now we apply $\neg\mathbb{F}$ to line 3:

1.	$\mathbb{F}\, \exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\, \exists x\, \neg\varphi(x)$ ✓	$\to\mathbb{F}\, 1$
3.	$\mathbb{F}\, \neg\forall x\, \varphi(x)$ ✓	$\to\mathbb{F}\, 1$
4.	$\mathbb{T}\, \neg\varphi(a)$	$\exists\mathbb{T}\, 2$
5.	$\mathbb{T}\, \forall x\, \varphi(x)$	$\neg\mathbb{F}\, 3$

We obtain a closed tableau by applying $\neg\mathbb{T}$ to line 4, followed by $\forall\mathbb{T}$ to line 5.

1.	$\mathbb{F}\, \exists x\, \neg\varphi(x) \to \neg\forall x\, \varphi(x)$ ✓	Assumption
2.	$\mathbb{T}\, \exists x\, \neg\varphi(x)$ ✓	$\to\mathbb{F}\, 1$
3.	$\mathbb{F}\, \neg\forall x\, \varphi(x)$ ✓	$\to\mathbb{F}\, 1$
4.	$\mathbb{T}\, \neg\varphi(a)$	$\exists\mathbb{T}\, 2$
5.	$\mathbb{T}\, \forall x\, \varphi(x)$	$\neg\mathbb{F}\, 3$
6.	$\mathbb{F}\, \varphi(a)$	$\neg\mathbb{T}\, 4$
7.	$\mathbb{T}\, \varphi(a)$	$\forall\mathbb{T}\, 5$
	$\otimes$	

Example tab.2. Let's see how we'd give a tableau for the set

$$\mathbb{F}\,\exists x\,\chi(x,b), \mathbb{T}\,\exists x\,(\varphi(x)\wedge\psi(x)), \mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b)).$$

Starting as usual, we start with the assumptions:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\wedge\psi(x))$	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption

We should always apply a rule with the eigenvariable condition first; in this case that would be $\exists\mathbb{T}$ to line 2. Since the assumptions contain the constant symbol b, we have to use a different one; let's pick a again.

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\wedge\psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a)\wedge\psi(a)$	$\exists\mathbb{T}\,2$

If we now apply $\exists\mathbb{F}$ to line 1 or $\forall\mathbb{T}$ to line 3, we have to decide which term t to substitute for x. Since there is no eigenvariable condition for these rules, we can pick any term we like. In some cases we may even have to apply the rule several times with different ts. But as a general rule, it pays to pick one of the terms already occurring in the tableau—in this case, a and b—and in this case we can guess that a will be more likely to result in a closed branch.

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\wedge\psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a)\wedge\psi(a)$	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a)\to\chi(a,b)$	$\forall\mathbb{T}\,3$

We don't check the signed formulas in lines 1 and 3, since we may have to use them again. Now apply $\wedge\mathbb{T}$ to line 4:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x)\wedge\psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x)\to\chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a)\wedge\psi(a)$ ✓	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a)\to\chi(a,b)$	$\forall\mathbb{T}\,3$
7.	$\mathbb{T}\,\varphi(a)$	$\wedge\mathbb{T}\,4$
8.	$\mathbb{T}\,\psi(a)$	$\wedge\mathbb{T}\,4$

If we now apply $\to\mathbb{T}$ to line 6, the tableau closes:

1.	$\mathbb{F}\,\exists x\,\chi(x,b)$	Assumption
2.	$\mathbb{T}\,\exists x\,(\varphi(x) \wedge \psi(x))$ ✓	Assumption
3.	$\mathbb{T}\,\forall x\,(\psi(x) \to \chi(x,b))$	Assumption
4.	$\mathbb{T}\,\varphi(a) \wedge \psi(a)$ ✓	$\exists\mathbb{T}\,2$
5.	$\mathbb{F}\,\chi(a,b)$	$\exists\mathbb{F}\,1$
6.	$\mathbb{T}\,\psi(a) \to \chi(a,b)$ ✓	$\forall\mathbb{T}\,3$
7.	$\mathbb{T}\,\varphi(a)$	$\wedge\mathbb{T}\,4$
8.	$\mathbb{T}\,\psi(a)$	$\wedge\mathbb{T}\,4$
9.	$\mathbb{F}\,\psi(a)$ $\quad$ $\mathbb{T}\,\chi(a,b)$ $\otimes$ $\quad$ $\otimes$	$\to\mathbb{T}\,6$

Example tab.3. We construct a tableau for the set

$$\mathbb{T}\,\forall x\,\varphi(x), \mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y), \mathbb{T}\,\neg\exists y\,\psi(y).$$

Starting as usual, we write down the assumptions:

1.	$\mathbb{T}\,\forall x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y)$	Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$	Assumption

We begin by applying the $\neg\mathbb{T}$ rule to line 3. A corollary to the rule "always apply rules with eigenvariable conditions first" is "defer applying quantifier rules without eigenvariable conditions until needed." Also, defer rules that result in a split.

1.	$\mathbb{T}\,\forall x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y)$	Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$ ✓	Assumption
4.	$\mathbb{F}\,\exists y\,\psi(y)$	$\neg\mathbb{T}\,3$

The new line 4 requires $\exists\mathbb{F}$, a quantifier rule without the eigenvariable condition. So we defer this in favor of using $\to\mathbb{T}$ on line 2.

1.	$\mathbb{T}\,\forall x\,\varphi(x)$	Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y)$ ✓	Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$ ✓	Assumption
4.	$\mathbb{F}\,\exists y\,\psi(y)$	$\neg\mathbb{T}\,3$
5.	$\mathbb{F}\,\forall x\,\varphi(x)$ $\quad$ $\mathbb{T}\,\exists y\,\psi(y)$	$\to\mathbb{T}\,2$

Both new signed formulas require rules with eigenvariable conditions, so these should be next:

1.	$\mathbb{T}\,\forall x\,\varphi(x)$		Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y)$ ✓		Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$ ✓		Assumption
4.	$\mathbb{F}\,\exists y\,\psi(y)$		$\neg\mathbb{T}\,3$
5.	$\mathbb{F}\,\forall x\,\varphi(x)$ ✓	$\mathbb{T}\,\exists y\,\psi(y)$ ✓	$\to\mathbb{T}\,2$
6.	$\mathbb{F}\,\varphi(b)$	$\mathbb{T}\,\psi(c)$	$\forall\mathbb{F}\,5$; $\exists\mathbb{T}\,5$

To close the branches, we have to use the signed formulas on lines 1 and 3. The corresponding rules ($\forall\mathbb{T}$ and $\exists\mathbb{F}$) don't have eigenvariable conditions, so we are free to pick whichever terms are suitable. In this case, that's b and c, respectively.

1.	$\mathbb{T}\,\forall x\,\varphi(x)$		Assumption
2.	$\mathbb{T}\,\forall x\,\varphi(x) \to \exists y\,\psi(y)$ ✓		Assumption
3.	$\mathbb{T}\,\neg\exists y\,\psi(y)$ ✓		Assumption
4.	$\mathbb{F}\,\exists y\,\psi(y)$		$\neg\mathbb{T}\,3$
5.	$\mathbb{F}\,\forall x\,\varphi(x)$ ✓	$\mathbb{T}\,\exists y\,\psi(y)$ ✓	$\to\mathbb{T}\,2$
6.	$\mathbb{F}\,\varphi(b)$	$\mathbb{T}\,\psi(c)$	$\forall\mathbb{F}\,5$; $\exists\mathbb{T}\,5$
7.	$\mathbb{T}\,\varphi(b)$	$\mathbb{F}\,\psi(c)$	$\forall\mathbb{T}\,1$; $\exists\mathbb{F}\,4$
	$\otimes$	$\otimes$	

Problem tab.1. Give closed tableaux of the following:

1. $\mathbb{F}\,(\forall x\,\varphi(x) \land \forall y\,\psi(y)) \to \forall z\,(\varphi(z) \land \psi(z))$.
2. $\mathbb{F}\,(\exists x\,\varphi(x) \lor \exists y\,\psi(y)) \to \exists z\,(\varphi(z) \lor \psi(z))$.
3. $\mathbb{T}\,\forall x\,(\varphi(x) \to \psi), \mathbb{F}\,\exists y\,\varphi(y) \to \psi$.
4. $\mathbb{T}\,\forall x\,\neg\varphi(x), \mathbb{F}\,\neg\exists x\,\varphi(x)$.
5. $\mathbb{F}\,\neg\exists x\,\varphi(x) \to \forall x\,\neg\varphi(x)$.
6. $\mathbb{F}\,\neg\exists x\,\forall y\,((\varphi(x,y) \to \neg\varphi(y,y)) \land (\neg\varphi(y,y) \to \varphi(x,y)))$.

Problem tab.2. Give closed tableaux of the following:

1. $\mathbb{F}\,\neg\forall x\,\varphi(x) \to \exists x\,\neg\varphi(x)$.
2. $\mathbb{T}\,(\forall x\,\varphi(x) \to \psi), \mathbb{F}\,\exists y\,(\varphi(y) \to \psi)$.
3. $\mathbb{F}\,\exists x\,(\varphi(x) \to \forall y\,\varphi(y))$.

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