tab.1 Derivability and the Propositional Connectives

fol:tab:ppr: sec We establish that the derivability relation $\vdash$ of tableaux is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \to \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. *explanation*

fol:tab:ppr: prop:provability-land **Proposition tab.1.**

fol:tab:ppr: prop:provability-land-left 1. *Both $\varphi \land \psi \vdash \varphi$ and $\varphi \land \psi \vdash \psi$.*

fol:tab:ppr: prop:provability-land-right 2. *$\varphi, \psi \vdash \varphi \land \psi$.*

Proof. 1. Both $\{\mathbb{F}\,\varphi, \mathbb{T}\,\varphi \land \psi\}$ and $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \land \psi\}$ have closed tableaux

$$\begin{array}{lcl}
1. & \mathbb{F}\,\varphi & \text{Assumption} \\
2. & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & \otimes &
\end{array}$$

$$\begin{array}{lcl}
1. & \mathbb{F}\,\psi & \text{Assumption} \\
2. & \mathbb{T}\,\varphi \land \psi & \text{Assumption} \\
3. & \mathbb{T}\varphi & \land\mathbb{T}\,2 \\
4. & \mathbb{T}\psi & \land\mathbb{T}\,2 \\
 & \otimes &
\end{array}$$

2. Here is a closed tableau for $\{\mathbb{T}\,\varphi, \mathbb{T}\,\psi, \mathbb{F}\,\varphi \land \psi\}$:

$$\begin{array}{lccl}
1. & \multicolumn{2}{c}{\mathbb{F}\,\varphi \land \psi} & \text{Assumption} \\
2. & \multicolumn{2}{c}{\mathbb{T}\,\varphi} & \text{Assumption} \\
3. & \multicolumn{2}{c}{\mathbb{T}\,\psi} & \text{Assumption} \\
4. & \mathbb{F}\,\varphi & \mathbb{F}\,\psi & \land\mathbb{F}\,1 \\
 & \otimes & \otimes &
\end{array}$$

fol:tab:ppr: prop:provability-lor **Proposition tab.2.**

1. *$\{\varphi \lor \psi, \lnot\varphi, \lnot\psi\}$ is inconsistent.*

2. *Both $\varphi \vdash \varphi \lor \psi$ and $\psi \vdash \varphi \lor \psi$.*

Proof. 1. We give a closed tableau of $\{\mathbb{T}\,\varphi \lor \psi, \mathbb{T}\,\lnot\varphi, \mathbb{T}\,\lnot\psi\}$:

$$
\begin{array}{lccl}
1. & & \mathbb{T}\,\varphi \lor \psi & \text{Assumption} \\
2. & & \mathbb{T}\,\neg\varphi & \text{Assumption} \\
3. & & \mathbb{T}\,\neg\psi & \text{Assumption} \\
4. & & \mathbb{F}\,\varphi & \neg\mathbb{T}\,2 \\
5. & & \mathbb{F}\,\psi & \neg\mathbb{T}\,3 \\
6. & \mathbb{T}\,\varphi & \mathbb{T}\,\psi & \lor\mathbb{T}\,1 \\
 & \otimes & \otimes &
\end{array}
$$

2. Both $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\varphi\}$ and $\{\mathbb{F}\,\varphi \lor \psi, \mathbb{T}\,\psi\}$ have closed tableaux:

$$
\begin{array}{lcl}
1. & \mathbb{F}\,\varphi \lor \psi & \text{Assumption} \\
2. & \mathbb{T}\,\varphi & \text{Assumption} \\
3. & \mathbb{F}\varphi & \lor\mathbb{F}\,1 \\
4. & \mathbb{F}\psi & \lor\mathbb{F}\,1 \\
 & \otimes &
\end{array}
$$

$$
\begin{array}{lcl}
1. & \mathbb{F}\,\varphi \lor \psi & \text{Assumption} \\
2. & \mathbb{T}\,\psi & \text{Assumption} \\
3. & \mathbb{F}\varphi & \lor\mathbb{F}\,1 \\
4. & \mathbb{F}\psi & \lor\mathbb{F}\,1 \\
 & \otimes &
\end{array}
$$

Proposition tab.3. *fol:tab:ppr: prop:provability-lif*

1. *$\varphi, \varphi \to \psi \vdash \psi$.* *fol:tab:ppr: prop:provability-lif-left*

2. *Both $\neg\varphi \vdash \varphi \to \psi$ and $\psi \vdash \varphi \to \psi$.* *fol:tab:ppr: prop:provability-lif-right*

Proof. 1. $\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi \to \psi, \mathbb{T}\,\varphi\}$ has a closed tableau:

$$
\begin{array}{lccl}
1. & & \mathbb{F}\,\psi & \text{Assumption} \\
2. & & \mathbb{T}\,\varphi \to \psi & \text{Assumption} \\
3. & & \mathbb{T}\,\varphi & \text{Assumption} \\
4. & \mathbb{F}\,\varphi & \mathbb{T}\,\psi & \to\mathbb{T}\,2 \\
 & \otimes & \otimes &
\end{array}
$$

2. Both $\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\neg\varphi\}$ and $\{\mathbb{F}\,\varphi \to \psi, \mathbb{T}\,\psi\}$ have closed tableaux:

1.	$\mathbb{F}\,\varphi \to \psi$	Assumption
2.	$\mathbb{T}\,\neg\varphi$	Assumption
3.	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,1$
5.	$\mathbb{F}\,\varphi$	$\neg\mathbb{T}\,2$
	$\otimes$	

1.	$\mathbb{F}\,\varphi \to \psi$	Assumption
2.	$\mathbb{T}\,\psi$	Assumption
3.	$\mathbb{T}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{F}\,\psi$	$\to\mathbb{F}\,1$
	$\otimes$	

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