

tab.1 Derivability and Consistency

fol:tab:prv:
sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:tab:prv:
prop:provability-contr **Proposition tab.1.** *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.*

Proof. There are finite $\Gamma_0 = \{\psi_1, \dots, \psi_n\}$ and $\Gamma_1 = \{\chi_1, \dots, \chi_m\} \subseteq \Gamma$ such that

$$\begin{aligned} &\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\} \\ &\{\mathbb{T} \varphi, \mathbb{T} \chi_1, \dots, \mathbb{T} \chi_m\} \end{aligned}$$

have closed **tableaux**. Using the Cut rule on φ we can combine these into a single closed **tableau** that shows $\Gamma_0 \cup \Gamma_1$ is inconsistent. Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. \square

fol:tab:prv:
prop:prov-incons **Proposition tab.2.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a closed **tableau** for

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

Using the $\neg\mathbb{T}$ rule, this can be turned into a closed **tableau** for

$$\{\mathbb{T} \neg\varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}.$$

On the other hand, if there is a closed **tableau** for the latter, we can turn it into a closed **tableau** of the former by removing every formula that results from $\neg\mathbb{T}$ applied to the first assumption $\mathbb{T} \neg\varphi$ as well as that assumption, and adding the assumption $\mathbb{F} \varphi$. For if a branch was closed before because it contained the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{T} \neg\varphi$, i.e., $\mathbb{F} \varphi$, the corresponding branch in the new **tableau** is also closed. If a branch in the old tableau was closed because it contained the assumption $\mathbb{T} \neg\varphi$ as well as $\mathbb{F} \neg\varphi$ we can turn it into a closed branch by applying $\neg\mathbb{F}$ to $\mathbb{F} \neg\varphi$ to obtain $\mathbb{T} \varphi$. This closes the branch since we added $\mathbb{F} \varphi$ as an assumption. \square

Problem tab.1. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:tab:prv:
prop:explicit-inc **Proposition tab.3.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there are $\psi_1, \dots, \psi_n \in \Gamma$ such that

$$\{\mathbb{F} \varphi, \mathbb{T} \psi_1, \dots, \mathbb{T} \psi_n\}$$

has a closed tableau. Replace the assumption $\mathbb{F} \varphi$ by $\mathbb{T} \neg\varphi$, and insert the conclusion of $\neg\mathbb{T}$ applied to $\mathbb{F} \varphi$ after the assumptions. Any **sentence** in the **tableau** justified by appeal to line 1 in the old **tableau** is now justified by appeal to line $n+1$. So if the old **tableau** was closed, the new one is. It shows that Γ is inconsistent, since all assumptions are in Γ . \square

Proposition tab.4. *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.* fol.tab:prv:
prop:provability-exhaustive

Proof. If there are $\psi_1, \dots, \psi_n \in \Gamma$ and $\chi_1, \dots, \chi_m \in \Gamma$ such that

$$\{\mathbb{T}\varphi, \mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n\} \text{ and} \\ \{\mathbb{T}\neg\varphi, \mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m\}$$

both have closed **tableaux**, we can construct a single, combined **tableau** that shows that Γ is inconsistent by using as assumptions $\mathbb{T}\psi_1, \dots, \mathbb{T}\psi_n$ together with $\mathbb{T}\chi_1, \dots, \mathbb{T}\chi_m$, followed by an application of the Cut rule. This yields two branches, one starting with $\mathbb{T}\varphi$, the other with $\mathbb{F}\varphi$.

On the left side, add the part of the first **tableau** below its assumptions. Here, every rule application is still correct, since each of the assumptions of the first **tableau**, including $\mathbb{T}\varphi$, is available. Thus, every branch below $\mathbb{T}\varphi$ closes.

On the right side, add the part of the second **tableau** below its assumption, with the results of any applications of $\neg\mathbb{T}$ to $\mathbb{T}\neg\varphi$ removed. The conclusion of $\neg\mathbb{T}$ to $\mathbb{T}\neg\varphi$ is $\mathbb{F}\varphi$, which is nevertheless available, as it is the conclusion of the Cut rule on the right side of the combined **tableau**.

If a branch in the second tableau was closed because it contained the assumption $\mathbb{T}\neg\varphi$ (which no longer appears as an assumption in the combined **tableau**) as well as $\mathbb{F}\neg\varphi$, we can applying $\neg\mathbb{F}$ to $\mathbb{F}\neg\varphi$ to obtain $\mathbb{T}\varphi$. Now the corresponding branch in the combined **tableau** also closes, because it contains the right-hand conclusion of the Cut rule, $\mathbb{F}\varphi$. If a branch in the second **tableau** closed for any other reason, the corresponding branch in the combined **tableau** also closes, since any **signed formulas** other than $\mathbb{T}\neg\varphi$ occurring on the branch in the old, second **tableau** also occur on the corresponding branch in the combined **tableau**. □

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Bibliography