tab.1 Proof-Theoretic Notions

fol:tab:ptn:
sec

> This section collects the definitions of the provability relation and consistency for tableaux.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the existence of certain closed tableaux. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. explanation

Definition tab.1 (Theorems). A sentence φ is a *theorem* if there is a closed tableau for $\mathbb{F}\,\varphi$. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition tab.2 (Derivability). A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$ iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Definition tab.3 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ and a closed tableau for the set

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

If Γ is not inconsistent, we say it is *consistent*.

fol:tab:ptn:
prop:reflexivity

Proposition tab.4 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. If $\varphi \in \Gamma$, $\{\varphi\}$ is a finite subset of Γ and the tableau

1.	$\mathbb{F}\,\varphi$	Assumption
2.	$\mathbb{T}\,\varphi$	Assumption
	$\otimes$	

is closed. □

fol:tab:ptn:
prop:monotonicity

Proposition tab.5 (Monotonicity). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any finite subset of Γ is also a finite subset of Δ. □

fol:tab:ptn:
prop:transitivity

Proposition tab.6 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a finite subset $\Delta_0 = \{\chi_1, \ldots, \chi_n\} \subseteq \Delta$ such that

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_n\}$$

has a closed tableau. If $\Gamma \vdash \varphi$ then there are θ_1, …, θ_m such that

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \ldots, \mathbb{T}\,\theta_m\}$$

has a closed tableau.

Now consider the tableau with assumptions

$$\mathbb{F}\,\psi, \mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_n, \mathbb{T}\,\theta_1, \ldots, \mathbb{T}\,\theta_m.$$

Apply the Cut rule on φ. This generates two branches, one has $\mathbb{T}\,\varphi$ in it, the other $\mathbb{F}\,\varphi$. Thus, on the one branch, all of

$$\{\mathbb{F}\,\psi, \mathbb{T}\,\varphi, \mathbb{T}\,\chi_1, \ldots, \mathbb{T}\,\chi_n\}$$

are available. Since there is a closed tableau for these assumptions, we can attach it to that branch; every branch through $\mathbb{T}\,\varphi$ closes. On the other branch, all of

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\theta_1, \ldots, \mathbb{T}\,\theta_m\}$$

are available, so we can also complete the other side to obtain a closed tableau. This shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

Proposition tab.7. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.* *fol:tab:ptn: prop:incons*

Proof. Exercise. □

Problem tab.1. Prove Proposition tab.7

Proposition tab.8 (Compactness). *fol:tab:ptn: prop:proves-compact*

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*
2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof.

1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 = \{\psi_1, \ldots, \psi_n\}$ and a closed tableau for
$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n\}$$
This tableau also shows $\Gamma_0 \vdash \varphi$.
2. If Γ is inconsistent, then for some finite subset $\Gamma_0 = \{\psi_1, \ldots, \psi_n\}$ there is a closed tableau for
$$\{\mathbb{T}\,\psi_1, \ldots, \mathbb{T}\,\psi_n\}$$
This closed tableau shows that Γ_0 is inconsistent. □

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Bibliography