tab.1 Tableaux with Identity predicate

fol:tab:ide:sec Tableaux with identity predicate require additional inference rules. The rules for $=$ are (t, t_1, and t_2 are closed terms):

$$\frac{}{\mathbb{T}\, t = t} =
\qquad
\frac{\begin{array}{c}\mathbb{T}\, t_1 = t_2 \\ \mathbb{T}\, \varphi(t_1)\end{array}}{\mathbb{T}\, \varphi(t_2)} =\mathbb{T}
\qquad
\frac{\begin{array}{c}\mathbb{T}\, t_1 = t_2 \\ \mathbb{F}\, \varphi(t_1)\end{array}}{\mathbb{F}\, \varphi(t_2)} =\mathbb{F}$$

Note that in contrast to all the other rules, $=\mathbb{T}$ and $=\mathbb{F}$ require that *two* signed formulas already appear on the branch, namely both $\mathbb{T}\, t_1 = t_2$ and $S\, \varphi(t_1)$.

Example tab.1. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

1.	$\mathbb{F}\, \varphi(t)$	Assumption
2.	$\mathbb{T}\, s = t$	Assumption
3.	$\mathbb{T}\, \varphi(s)$	Assumption
4.	$\mathbb{T}\, \varphi(t)$	$=\mathbb{T}\, 2, 3$
	$\otimes$	

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

Tableaux prove that $=$ is symmetric, i.e., that $s_1 = s_2 \vdash s_2 = s_1$:

1.	$\mathbb{F}\, s_2 = s_1$	Assumption
2.	$\mathbb{T}\, s_1 = s_2$	Assumption
3.	$\mathbb{T}\, s_1 = s_1$	$=$
4.	$\mathbb{T}\, s_2 = s_1$	$=\mathbb{T}\, 2, 3$
	$\otimes$	

Here, line 2 is the first prerequisite formula $\mathbb{T}\, s_1 = s_2$ of $=\mathbb{T}$. Line 3 is the second one, of the form $\mathbb{T}\, \varphi(s_2)$—think of $\varphi(x)$ as $x = s_1$, then $\varphi(s_1)$ is $s_1 = s_1$ and $\varphi(s_2)$ is $s_2 = s_1$.

They also prove that $=$ is transitive, i.e., that $s_1 = s_2, s_2 = s_3 \vdash s_1 = s_3$:

1.	$\mathbb{F}\, s_1 = s_3$	Assumption
2.	$\mathbb{T}\, s_1 = s_2$	Assumption
3.	$\mathbb{T}\, s_2 = s_3$	Assumption
4.	$\mathbb{T}\, s_1 = s_3$	$=\mathbb{T}\, 3, 2$
	$\otimes$	

In this tableau, the first prerequisite formula of $=\mathbb{T}$ is line 3, $\mathbb{T}\, s_2 = s_3$ (s_2 plays the role of t_1, and s_3 the role of t_2). The second prerequisite, of the form $\mathbb{T}\,\varphi(s_2)$ is line 2. Here, think of $\varphi(x)$ as $s_1 = x$; that makes $\varphi(s_2)$ into $t_1 = t_2$ (i.e., line 2) and $\varphi(s_3)$ into the formula $s_1 = s_3$ in the conclusion.

Problem tab.1. Give closed tableaux for the following:

1. $\mathbb{F}\,\forall x\,\forall y\,((x = y \land \varphi(x)) \rightarrow \varphi(y))$
2. $\mathbb{F}\,\exists x\,(\varphi(x) \land \forall y\,(\varphi(y) \rightarrow y = x))$,
   $\mathbb{T}\,\exists x\,\varphi(x) \land \forall y\,\forall z\,((\varphi(y) \land \varphi(z)) \rightarrow y = z)$

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Bibliography