

syn.1 Substitution

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Definition syn.1 (Substitution in a term). We define $s[t/x]$, the result of substituting t for every occurrence of x in s , recursively:

1. $s \equiv c$: $s[t/x]$ is just s .
2. $s \equiv y$: $s[t/x]$ is also just s , provided y is a variable and $y \neq x$.
3. $s \equiv x$: $s[t/x]$ is t .
4. $s \equiv f(t_1, \dots, t_n)$: $s[t/x]$ is $f(t_1[t/x], \dots, t_n[t/x])$.

Definition syn.2. A term t is *free for* x in φ if none of the free occurrences of x in φ occur in the scope of a quantifier that binds a variable in t .

Example syn.3.

1. v_8 is free for v_1 in $\exists v_3 A_4^2(v_3, v_1)$
2. $f_1^2(v_1, v_2)$ is *not* free for v_0 in $\forall v_2 A_4^2(v_0, v_2)$

Definition syn.4 (Substitution in a formula). If φ is a formula, x is a variable, and t is a term free for x in φ , then $\varphi[t/x]$ is the result of substituting t for all free occurrences of x in φ .

1. $\varphi \equiv \perp$: $\varphi[t/x]$ is \perp .
2. $\varphi \equiv \top$: $\varphi[t/x]$ is \top .
3. $\varphi \equiv P(t_1, \dots, t_n)$: $\varphi[t/x]$ is $P(t_1[t/x], \dots, t_n[t/x])$.
4. $\varphi \equiv t_1 = t_2$: $\varphi[t/x]$ is $t_1[t/x] = t_2[t/x]$.
5. $\varphi \equiv \neg\psi$: $\varphi[t/x]$ is $\neg\psi[t/x]$.
6. $\varphi \equiv (\psi \wedge \chi)$: $\varphi[t/x]$ is $(\psi[t/x] \wedge \chi[t/x])$.
7. $\varphi \equiv (\psi \vee \chi)$: $\varphi[t/x]$ is $(\psi[t/x] \vee \chi[t/x])$.
8. $\varphi \equiv (\psi \rightarrow \chi)$: $\varphi[t/x]$ is $(\psi[t/x] \rightarrow \chi[t/x])$.
9. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\varphi[t/x]$ is $(\psi[t/x] \leftrightarrow \chi[t/x])$.
10. $\varphi \equiv \forall y \psi$: $\varphi[t/x]$ is $\forall y \psi[t/x]$, provided y is a variable other than x ; otherwise $\varphi[t/x]$ is just φ .
11. $\varphi \equiv \exists y \psi$: $\varphi[t/x]$ is $\exists y \psi[t/x]$, provided y is a variable other than x ; otherwise $\varphi[t/x]$ is just φ .

explanation Note that substitution may be vacuous: If x does not occur in φ at all, then $\varphi[t/x]$ is just φ .

The restriction that t must be **free for x** in φ is necessary to exclude cases like the following. If $\varphi \equiv \exists y x < y$ and $t \equiv y$, then $\varphi[t/x]$ would be $\exists y y < y$. In this case the free variable y is “captured” by the quantifier $\exists y$ upon substitution, and that is undesirable. For instance, we would like it to be the case that whenever $\forall x \psi$ holds, so does $\psi[t/x]$. But consider $\forall x \exists y x < y$ (here ψ is $\exists y x < y$). It is a sentence that is true about, e.g., the natural numbers: for every number x there is a number y greater than it. If we allowed y as a possible substitution for x , we would end up with $\psi[y/x] \equiv \exists y y < y$, which is false. We prevent this by requiring that none of the free variables in t would end up being bound by a quantifier in φ .

We often use the following convention to avoid cumbersome notation: If φ is a **formula** which may contain the **variable x** free, we also write $\varphi(x)$ to indicate this. When it is clear which φ and x we have in mind, and t is a term (assumed to be free for x in $\varphi(x)$), then we write $\varphi(t)$ as short for $\varphi[t/x]$. So for instance, we might say, “we call $\varphi(t)$ an instance of $\forall x \varphi(x)$.” By this we mean that if φ is any **formula**, x a **variable**, and t a term that’s free for x in φ , then $\varphi[t/x]$ is an instance of $\forall x \varphi$.

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Bibliography