

## syn.1 Semantic Notions

fol:syn:sem:  
sec

Given the definition of **structures** for first-order languages, we can define some basic semantic properties of and relationships between sentences. The simplest of these is the notion of *validity* of a sentence. A sentence is valid if it is satisfied in every **structure**. Valid sentences are those that are satisfied regardless of how the non-logical symbols in it are interpreted. Valid sentences are therefore also called *logical truths*—they are true, i.e., satisfied, in any **structure** and hence their truth depends only on the logical symbols occurring in them and their syntactic **structure**, but not on the non-logical symbols or their interpretation.

explanation

**Definition syn.1 (Validity).** A sentence  $\varphi$  is *valid*,  $\models \varphi$ , iff  $\mathfrak{M} \models \varphi$  for every **structure**  $\mathfrak{M}$ .

**Definition syn.2 (Entailment).** A set of sentences  $\Gamma$  *entails* a sentence  $\varphi$ ,  $\Gamma \models \varphi$ , iff for every **structure**  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma$ ,  $\mathfrak{M} \models \varphi$ .

**Definition syn.3 (Satisfiability).** A set of sentences  $\Gamma$  is *satisfiable* if  $\mathfrak{M} \models \Gamma$  for some **structure**  $\mathfrak{M}$ . If  $\Gamma$  is not satisfiable it is called *unsatisfiable*.

**Proposition syn.4.** A sentence  $\varphi$  is valid iff  $\Gamma \models \varphi$  for every set of sentences  $\Gamma$ .

*Proof.* For the forward direction, let  $\varphi$  be valid, and let  $\Gamma$  be a set of sentences. Let  $\mathfrak{M}$  be a **structure** so that  $\mathfrak{M} \models \Gamma$ . Since  $\varphi$  is valid,  $\mathfrak{M} \models \varphi$ , hence  $\Gamma \models \varphi$ .

For the contrapositive of the reverse direction, let  $\varphi$  be invalid, so there is a **structure**  $\mathfrak{M}$  with  $\mathfrak{M} \not\models \varphi$ . When  $\Gamma = \{\top\}$ , since  $\top$  is valid,  $\mathfrak{M} \models \Gamma$ . Hence, there is a **structure**  $\mathfrak{M}$  so that  $\mathfrak{M} \models \Gamma$  but  $\mathfrak{M} \not\models \varphi$ , hence  $\Gamma$  does not entail  $\varphi$ .  $\square$

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prop:entails-unsat

**Proposition syn.5.**  $\Gamma \models \varphi$  iff  $\Gamma \cup \{\neg\varphi\}$  is unsatisfiable.

*Proof.* For the forward direction, suppose  $\Gamma \models \varphi$  and suppose to the contrary that there is a **structure**  $\mathfrak{M}$  so that  $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$ . Since  $\mathfrak{M} \models \Gamma$  and  $\Gamma \models \varphi$ ,  $\mathfrak{M} \models \varphi$ . Also, since  $\mathfrak{M} \models \Gamma \cup \{\neg\varphi\}$ ,  $\mathfrak{M} \models \neg\varphi$ , so we have both  $\mathfrak{M} \models \varphi$  and  $\mathfrak{M} \models \neg\varphi$ , a contradiction. Hence, there can be no such **structure**  $\mathfrak{M}$ , so  $\Gamma \cup \{\neg\varphi\}$  is unsatisfiable.

For the reverse direction, suppose  $\Gamma \cup \{\neg\varphi\}$  is unsatisfiable. So for every **structure**  $\mathfrak{M}$ , either  $\mathfrak{M} \not\models \Gamma$  or  $\mathfrak{M} \models \varphi$ . Hence, for every **structure**  $\mathfrak{M}$  with  $\mathfrak{M} \models \Gamma$ ,  $\mathfrak{M} \models \varphi$ , so  $\Gamma \models \varphi$ .  $\square$

**Problem syn.1.** 1. Show that  $\Gamma \models \perp$  iff  $\Gamma$  is unsatisfiable.

2. Show that  $\Gamma \cup \{\varphi\} \models \perp$  iff  $\Gamma \models \neg\varphi$ .

3. Suppose  $c$  does not occur in  $\varphi$  or  $\Gamma$ . Show that  $\Gamma \models \forall x \varphi$  iff  $\Gamma \models \varphi[c/x]$ .

**Proposition syn.6.** If  $\Gamma \subseteq \Gamma'$  and  $\Gamma \models \varphi$ , then  $\Gamma' \models \varphi$ .

*Proof.* Suppose that  $\Gamma \subseteq \Gamma'$  and  $\Gamma \models \varphi$ . Let  $\mathfrak{M}$  be a structure such that  $\mathfrak{M} \models \Gamma'$ ; then  $\mathfrak{M} \models \Gamma$ , and since  $\Gamma \models \varphi$ , we get that  $\mathfrak{M} \models \varphi$ . Hence, whenever  $\mathfrak{M} \models \Gamma'$ ,  $\mathfrak{M} \models \varphi$ , so  $\Gamma' \models \varphi$ .  $\square$

**Theorem syn.7 (Semantic Deduction Theorem).**  $\Gamma \cup \{\varphi\} \models \psi$  iff  $\Gamma \models \varphi \rightarrow \psi$ . fol:syn:sem:  
thm:sem-deduction

*Proof.* For the forward direction, let  $\Gamma \cup \{\varphi\} \models \psi$  and let  $\mathfrak{M}$  be a structure so that  $\mathfrak{M} \models \Gamma$ . If  $\mathfrak{M} \models \varphi$ , then  $\mathfrak{M} \models \Gamma \cup \{\varphi\}$ , so since  $\Gamma \cup \{\varphi\}$  entails  $\psi$ , we get  $\mathfrak{M} \models \psi$ . Therefore,  $\mathfrak{M} \models \varphi \rightarrow \psi$ , so  $\Gamma \models \varphi \rightarrow \psi$ .

For the reverse direction, let  $\Gamma \models \varphi \rightarrow \psi$  and  $\mathfrak{M}$  be a structure so that  $\mathfrak{M} \models \Gamma \cup \{\varphi\}$ . Then  $\mathfrak{M} \models \Gamma$ , so  $\mathfrak{M} \models \varphi \rightarrow \psi$ , and since  $\mathfrak{M} \models \varphi$ ,  $\mathfrak{M} \models \psi$ . Hence, whenever  $\mathfrak{M} \models \Gamma \cup \{\varphi\}$ ,  $\mathfrak{M} \models \psi$ , so  $\Gamma \cup \{\varphi\} \models \psi$ .  $\square$

**Proposition syn.8.** Let  $\mathfrak{M}$  be a structure, and  $\varphi(x)$  a formula with one free variable  $x$ , and  $t$  a closed term. Then: fol:syn:sem:  
prop:quant-terms

1.  $\varphi(t) \models \exists x \varphi(x)$
2.  $\forall x \varphi(x) \models \varphi(t)$

*Proof.* 1. Suppose  $\mathfrak{M} \models \varphi(t)$ . Let  $s$  be a variable assignment with  $s(x) = \text{Val}^{\mathfrak{M}}(t)$ . Then  $\mathfrak{M}, s \models \varphi(t)$  since  $\varphi(t)$  is a sentence. By ??,  $\mathfrak{M}, s \models \varphi(x)$ . By ??,  $\mathfrak{M} \models \exists x \varphi(x)$ .

2. Suppose  $\mathfrak{M} \models \forall x \varphi(x)$ . Let  $s$  be a variable assignment with  $s(x) = \text{Val}^{\mathfrak{M}}(t)$ . By ??,  $\mathfrak{M}, s \models \varphi(x)$ . By ??,  $\mathfrak{M}, s \models \varphi(t)$ . By ??,  $\mathfrak{M} \models \varphi(t)$  since  $\varphi(t)$  is a sentence.  $\square$

**Problem syn.2.** Complete the proof of **Proposition syn.8**.

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## Bibliography