

## syn.1 Satisfaction of a Formula in a Structure

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sec

The basic notion that relates expressions such as terms and formulas, on the one hand, and structures on the other, are those of *value* of a term and *satisfaction* of a formula. Informally, the *value* of a term is an *element* of a *structure*—if the term is just a constant, its *value* is the object assigned to the constant by the *structure*, and if it is built up using *function symbols*, the *value* is computed from the *values* of constants and the functions assigned to the functions in the term. A *formula* is *satisfied* in a *structure* if the interpretation given to the predicates makes the *formula* true in the domain of the *structure*. This notion of satisfaction is specified inductively: the specification of the *structure* directly states when atomic *formulas* are satisfied, and we define when a complex *formula* is satisfied depending on the main connective or quantifier and whether or not the immediate *subformulas* are satisfied.

explanation

The case of the quantifiers here is a bit tricky, as the immediate *subformula* of a quantified *formula* has a free *variable*, and *structures* don't specify the *values* of *variables*. In order to deal with this difficulty, we also introduce *variable assignments* and define satisfaction not with respect to a *structure* alone, but with respect to a *structure* plus a *variable* assignment.

**Definition syn.1 (Variable Assignment).** A *variable assignment*  $s$  for a *structure*  $\mathfrak{M}$  is a function which maps each *variable* to an *element* of  $|\mathfrak{M}|$ , i.e.,  $s: \text{Var} \rightarrow |\mathfrak{M}|$ .

A *structure* assigns a *value* to each *constant symbol*, and a *variable assignment* to each *variable*. But we want to use terms built up from them to also name *elements* of the *domain*. For this we define the *value* of terms inductively. For *constant symbols* and *variables* the *value* is just as the *structure* or the *variable assignment* specifies it; for more complex terms it is computed recursively using the functions the *structure* assigns to the *function symbols*.

explanation

**Definition syn.2 (Value of Terms).** If  $t$  is a term of the language  $\mathcal{L}$ ,  $\mathfrak{M}$  is a *structure* for  $\mathcal{L}$ , and  $s$  is a *variable assignment* for  $\mathfrak{M}$ , the *value*  $\text{Val}_s^{\mathfrak{M}}(t)$  is defined as follows:

1.  $t \equiv c: \text{Val}_s^{\mathfrak{M}}(t) = c^{\mathfrak{M}}$ .
2.  $t \equiv x: \text{Val}_s^{\mathfrak{M}}(t) = s(x)$ .
3.  $t \equiv f(t_1, \dots, t_n):$

$$\text{Val}_s^{\mathfrak{M}}(t) = f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)).$$

**Definition syn.3 ( $x$ -Variant).** If  $s$  is a *variable assignment* for a *structure*  $\mathfrak{M}$ , then any *variable assignment*  $s'$  for  $\mathfrak{M}$  which differs from  $s$  at most in what it assigns to  $x$  is called an  *$x$ -variant* of  $s$ . If  $s'$  is an  *$x$ -variant* of  $s$  we write  $s' \sim_x s$ .

explanation

Note that an  $x$ -variant of an assignment  $s$  does not *have* to assign something different to  $x$ . In fact, every assignment counts as an  $x$ -variant of itself.

**Definition syn.4.** If  $s$  is a **variable** assignment for a **structure**  $\mathfrak{M}$  and  $m \in |\mathfrak{M}|$ , then the assignment  $s[m/x]$  is the variable assignment defined by

$$s[m/x](y) = \begin{cases} m & \text{if } y \equiv x \\ s(y) & \text{otherwise.} \end{cases}$$

In other words,  $s[m/x]$  is the particular  $x$ -variant of  $s$  which assigns the domain **element**  $m$  to  $x$ , and assigns the same things to **variables** other than  $x$  that  $s$  does.

**Definition syn.5 (Satisfaction).** Satisfaction of a **formula**  $\varphi$  in a **structure**  $\mathfrak{M}$  relative to a **variable** assignment  $s$ , in symbols:  $\mathfrak{M}, s \models \varphi$ , is defined recursively as follows. (We write  $\mathfrak{M}, s \not\models \varphi$  to mean “not  $\mathfrak{M}, s \models \varphi$ .”)

fol:syn:sat:  
defn:satisfaction

1.  $\varphi \equiv \perp$ :  $\mathfrak{M}, s \not\models \varphi$ .
2.  $\varphi \equiv \top$ :  $\mathfrak{M}, s \models \varphi$ .
3.  $\varphi \equiv R(t_1, \dots, t_n)$ :  $\mathfrak{M}, s \models \varphi$  iff  $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n) \rangle \in R^{\mathfrak{M}}$ .
4.  $\varphi \equiv t_1 = t_2$ :  $\mathfrak{M}, s \models \varphi$  iff  $\text{Val}_s^{\mathfrak{M}}(t_1) = \text{Val}_s^{\mathfrak{M}}(t_2)$ .
5.  $\varphi \equiv \neg\psi$ :  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}, s \not\models \psi$ .
6.  $\varphi \equiv (\psi \wedge \chi)$ :  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}, s \models \psi$  and  $\mathfrak{M}, s \models \chi$ .
7.  $\varphi \equiv (\psi \vee \chi)$ :  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}, s \models \psi$  or  $\mathfrak{M}, s \models \chi$  (or both).
8.  $\varphi \equiv (\psi \rightarrow \chi)$ :  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}, s \not\models \psi$  or  $\mathfrak{M}, s \models \chi$  (or both).
9.  $\varphi \equiv (\psi \leftrightarrow \chi)$ :  $\mathfrak{M}, s \models \varphi$  iff either both  $\mathfrak{M}, s \models \psi$  and  $\mathfrak{M}, s \models \chi$ , or neither  $\mathfrak{M}, s \models \psi$  nor  $\mathfrak{M}, s \models \chi$ .
10.  $\varphi \equiv \forall x \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for every **element**  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M}, s[m/x] \models \psi$ .
11.  $\varphi \equiv \exists x \psi$ :  $\mathfrak{M}, s \models \varphi$  iff for at least one **element**  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M}, s[m/x] \models \psi$ .

explanation

The variable assignments are important in the last two clauses. We cannot define satisfaction of  $\forall x \psi(x)$  by “for all  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M} \models \psi(m)$ .” We cannot define satisfaction of  $\exists x \psi(x)$  by “for at least one  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M} \models \psi(m)$ .” The reason is that if  $m \in |\mathfrak{M}|$ , it is not a symbol of the language, and so  $\psi(m)$  is not a **formula** (that is,  $\psi[m/x]$  is undefined). We also cannot assume that we have **constant symbols** or terms available that name every **element** of  $\mathfrak{M}$ , since there is nothing in the definition of **structures** that requires it. In the standard language, the set of **constant symbols** is **denumerable**, so if  $|\mathfrak{M}|$  is not **enumerable** there aren’t even enough **constant symbols** to name every object.

We solve this problem by introducing **variable** assignments, which allow us to link variables directly with **elements** of the domain. Then instead of saying

that, e.g.,  $\exists x \psi(x)$  is satisfied in  $\mathfrak{M}$  iff for at least one  $m \in |\mathfrak{M}|$ , we say it is satisfied in  $\mathfrak{M}$  *relative to*  $s$  iff  $\psi(x)$  is satisfied relative to  $s[m/x]$  for at least one  $m \in |\mathfrak{M}|$ .

**Example syn.6.** Let  $\mathcal{L} = \{a, b, f, R\}$  where  $a$  and  $b$  are **constant symbols**,  $f$  is a two-place **function symbol**, and  $R$  is a two-place **predicate symbol**. Consider the **structure**  $\mathfrak{M}$  defined by:

1.  $|\mathfrak{M}| = \{1, 2, 3, 4\}$
2.  $a^{\mathfrak{M}} = 1$
3.  $b^{\mathfrak{M}} = 2$
4.  $f^{\mathfrak{M}}(x, y) = x + y$  if  $x + y \leq 3$  and  $= 3$  otherwise.
5.  $R^{\mathfrak{M}} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 2, 4 \rangle\}$

The function  $s(x) = 1$  that assigns  $1 \in |\mathfrak{M}|$  to every **variable** is a variable assignment for  $\mathfrak{M}$ .

Then

$$\text{Val}_s^{\mathfrak{M}}(f(a, b)) = f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(a), \text{Val}_s^{\mathfrak{M}}(b)).$$

Since  $a$  and  $b$  are **constant symbols**,  $\text{Val}_s^{\mathfrak{M}}(a) = a^{\mathfrak{M}} = 1$  and  $\text{Val}_s^{\mathfrak{M}}(b) = b^{\mathfrak{M}} = 2$ . So

$$\text{Val}_s^{\mathfrak{M}}(f(a, b)) = f^{\mathfrak{M}}(1, 2) = 1 + 2 = 3.$$

To compute the value of  $f(f(a, b), a)$  we have to consider

$$\text{Val}_s^{\mathfrak{M}}(f(f(a, b), a)) = f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(f(a, b)), \text{Val}_s^{\mathfrak{M}}(a)) = f^{\mathfrak{M}}(3, 1) = 3,$$

since  $3 + 1 > 3$ . Since  $s(x) = 1$  and  $\text{Val}_s^{\mathfrak{M}}(x) = s(x)$ , we also have

$$\text{Val}_s^{\mathfrak{M}}(f(f(a, b), x)) = f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(f(a, b)), \text{Val}_s^{\mathfrak{M}}(x)) = f^{\mathfrak{M}}(3, 1) = 3,$$

An atomic **formula**  $R(t_1, t_2)$  is satisfied if the tuple of values of its arguments, i.e.,  $\langle \text{Val}_s^{\mathfrak{M}}(t_1), \text{Val}_s^{\mathfrak{M}}(t_2) \rangle$ , is **an element** of  $R^{\mathfrak{M}}$ . So, e.g., we have  $\mathfrak{M}, s \models R(b, f(a, b))$  since  $\langle \text{Val}_s^{\mathfrak{M}}(b), \text{Val}_s^{\mathfrak{M}}(f(a, b)) \rangle = \langle 2, 3 \rangle \in R^{\mathfrak{M}}$ , but  $\mathfrak{M}, s \not\models R(x, f(a, b))$  since  $\langle 1, 3 \rangle \notin R^{\mathfrak{M}}[s]$ .

To determine if a non-atomic formula  $\varphi$  is satisfied, you apply the clauses in the inductive definition that applies to the main connective. For instance, the main connective in  $R(a, a) \rightarrow (R(b, x) \vee R(x, b))$  is the  $\rightarrow$ , and

$$\begin{aligned} \mathfrak{M}, s \models R(a, a) \rightarrow (R(b, x) \vee R(x, b)) \text{ iff} \\ \mathfrak{M}, s \not\models R(a, a) \text{ or } \mathfrak{M}, s \models R(b, x) \vee R(x, b) \end{aligned}$$

Since  $\mathfrak{M}, s \models R(a, a)$  (because  $\langle 1, 1 \rangle \in R^{\mathfrak{M}}$ ) we can't yet determine the answer and must first figure out if  $\mathfrak{M}, s \models R(b, x) \vee R(x, b)$ :

$$\begin{aligned} \mathfrak{M}, s \models R(b, x) \vee R(x, b) &\text{ iff} \\ \mathfrak{M}, s \models R(b, x) &\text{ or } \mathfrak{M}, s \models R(x, b) \end{aligned}$$

And this is the case, since  $\mathfrak{M}, s \models R(x, b)$  (because  $\langle 1, 2 \rangle \in R^{\mathfrak{M}}$ ).

Recall that an  $x$ -variant of  $s$  is a variable assignment that differs from  $s$  at most in what it assigns to  $x$ . For every **element** of  $|\mathfrak{M}|$ , there is an  $x$ -variant of  $s$ :

$$\begin{aligned} s_1 &= s[1/x], & s_2 &= s[2/x], \\ s_3 &= s[3/x], & s_4 &= s[4/x]. \end{aligned}$$

So, e.g.,  $s_2(x) = 2$  and  $s_2(y) = s(y) = 1$  for all variables  $y$  other than  $x$ . These are all the  $x$ -variants of  $s$  for the structure  $\mathfrak{M}$ , since  $|\mathfrak{M}| = \{1, 2, 3, 4\}$ . Note, in particular, that  $s_1 = s$  ( $s$  is always an  $x$ -variant of itself).

To determine if an existentially quantified **formula**  $\exists x \varphi(x)$  is satisfied, we have to determine if  $\mathfrak{M}, s[m/x] \models \varphi(x)$  for at least one  $m \in |\mathfrak{M}|$ . So,

$$\mathfrak{M}, s \models \exists x (R(b, x) \vee R(x, b)),$$

since  $\mathfrak{M}, s[1/x] \models R(b, x) \vee R(x, b)$  ( $s[3/x]$  would also fit the bill). But,

$$\mathfrak{M}, s \not\models \exists x (R(b, x) \wedge R(x, b))$$

since, whichever  $m \in |\mathfrak{M}|$  we pick,  $\mathfrak{M}, s[m/x] \not\models R(b, x) \wedge R(x, b)$ .

To determine if a universally quantified **formula**  $\forall x \varphi(x)$  is satisfied, we have to determine if  $\mathfrak{M}, s[m/x] \models \varphi(x)$  for all  $m \in |\mathfrak{M}|$ . So,

$$\mathfrak{M}, s \models \forall x (R(x, a) \rightarrow R(a, x)),$$

since  $\mathfrak{M}, s[m/x] \models R(x, a) \rightarrow R(a, x)$  for all  $m \in |\mathfrak{M}|$ . For  $m = 1$ , we have  $\mathfrak{M}, s[1/x] \models R(a, x)$  so the consequent is true; for  $m = 2, 3$ , and  $4$ , we have  $\mathfrak{M}, s[m/x] \not\models R(x, a)$ , so the antecedent is false. But,

$$\mathfrak{M}, s \not\models \forall x (R(a, x) \rightarrow R(x, a))$$

since  $\mathfrak{M}, s[2/x] \not\models R(a, x) \rightarrow R(x, a)$  (because  $\mathfrak{M}, s[2/x] \models R(a, x)$  and  $\mathfrak{M}, s[2/x] \not\models R(x, a)$ ).

For a more complicated case, consider

$$\forall x (R(a, x) \rightarrow \exists y R(x, y)).$$

Since  $\mathfrak{M}, s[3/x] \not\models R(a, x)$  and  $\mathfrak{M}, s[4/x] \not\models R(a, x)$ , the interesting cases where we have to worry about the consequent of the conditional are only  $m = 1$

and  $= 2$ . Does  $\mathfrak{M}, s[1/x] \models \exists y R(x, y)$  hold? It does if there is at least one  $n \in |\mathfrak{M}|$  so that  $\mathfrak{M}, s[1/x][n/y] \models R(x, y)$ . In fact, if we take  $n = 1$ , we have  $s[1/x][n/y] = s[1/y] = s$ . Since  $s(x) = 1$ ,  $s(y) = 1$ , and  $\langle 1, 1 \rangle \in R^{\mathfrak{M}}$ , the answer is yes.

To determine if  $\mathfrak{M}, s[2/x] \models \exists y R(x, y)$ , we have to look at the **variable** assignments  $s[2/x][n/y]$ . Here, for  $n = 1$ , this assignment is  $s_2 = s[2/x]$ , which does not satisfy  $R(x, y)$  ( $s_2(x) = 2$ ,  $s_2(y) = 1$ , and  $\langle 2, 1 \rangle \notin R^{\mathfrak{M}}$ ). However, consider  $s[2/x][3/y] = s_2[3/y]$ .  $\mathfrak{M}, s_2[3/y] \models R(x, y)$  since  $\langle 2, 3 \rangle \in R^{\mathfrak{M}}$ , and so  $\mathfrak{M}, s_2 \models \exists y R(x, y)$ .

So, for all  $n \in |\mathfrak{M}|$ , either  $\mathfrak{M}, s[m/x] \not\models R(a, x)$  (if  $m = 3, 4$ ) or  $\mathfrak{M}, s[m/x] \models \exists y R(x, y)$  (if  $m = 1, 2$ ), and so

$$\mathfrak{M}, s \models \forall x (R(a, x) \rightarrow \exists y R(x, y)).$$

On the other hand,

$$\mathfrak{M}, s \not\models \exists x (R(a, x) \wedge \forall y R(x, y)).$$

We have  $\mathfrak{M}, s[m/x] \models R(a, x)$  only for  $m = 1$  and  $m = 2$ . But for both of these values of  $m$ , there is in turn an  $n \in |\mathfrak{M}|$ , namely  $n = 4$ , so that  $\mathfrak{M}, s[m/x][n/y] \not\models R(x, y)$  and so  $\mathfrak{M}, s[m/x] \not\models \forall y R(x, y)$  for  $m = 1$  and  $m = 2$ . In sum, there is no  $m \in |\mathfrak{M}|$  such that  $\mathfrak{M}, s[m/x] \models R(a, x) \wedge \forall y R(x, y)$ .

**Problem syn.1.** Let  $\mathcal{L} = \{c, f, A\}$  with one **constant symbol**, one one-place **function symbol** and one two-place **predicate symbol**, and let the **structure**  $\mathfrak{M}$  be given by

1.  $|\mathfrak{M}| = \{1, 2, 3\}$
2.  $c^{\mathfrak{M}} = 3$
3.  $f^{\mathfrak{M}}(1) = 2, f^{\mathfrak{M}}(2) = 3, f^{\mathfrak{M}}(3) = 2$
4.  $A^{\mathfrak{M}} = \{\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle\}$

(a) Let  $s(v) = 1$  for all **variables**  $v$ . Find out whether

$$\mathfrak{M}, s \models \exists x (A(f(z), c) \rightarrow \forall y (A(y, x) \vee A(f(y), x)))$$

Explain why or why not.

(b) Give a different structure and **variable** assignment in which the **formula** is not satisfied.

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