syn.1 Extensionality

fol:syn:ext: sec

Extensionality, sometimes called relevance, can be expressed informally as follows: the only factors that bear upon the satisfaction of formula φ in a structure $\mathfrak{M}$ relative to a variable assignment s, are the size of the domain and the assignments made by $\mathfrak{M}$ and s to the elements of the language that actually appear in φ. explanation

One immediate consequence of extensionality is that where two structures $\mathfrak{M}$ and $\mathfrak{M}'$ agree on all the elements of the language appearing in a sentence φ and have the same domain, $\mathfrak{M}$ and $\mathfrak{M}'$ must also agree on whether or not φ itself is true.

fol:syn:ext: prop:extensionality

Proposition syn.1 (Extensionality). *Let φ be a formula, and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be structures with $|\mathfrak{M}_1| = |\mathfrak{M}_2|$, and s a variable assignment on $|\mathfrak{M}_1| = |\mathfrak{M}_2|$. If $c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}$, $R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2}$, and $f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}$ for every constant symbol c, relation symbol R, and function symbol f occurring in φ, then $\mathfrak{M}_1, s \vDash \varphi$ iff $\mathfrak{M}_2, s \vDash \varphi$.*

Proof. First prove (by induction on t) that for every term, $\mathrm{Val}^{\mathfrak{M}_1}_s(t) = \mathrm{Val}^{\mathfrak{M}_2}_s(t)$. Then prove the proposition by induction on φ, making use of the claim just proved for the induction basis (where φ is atomic). □

Problem syn.1. Carry out the proof of Proposition syn.1 in detail.

fol:syn:ext: cor:extensionality-sent

Corollary syn.2 (Extensionality for Sentences). *Let φ be a sentence and $\mathfrak{M}_1$, $\mathfrak{M}_2$ as in Proposition syn.1. Then $\mathfrak{M}_1 \vDash \varphi$ iff $\mathfrak{M}_2 \vDash \varphi$.*

Proof. Follows from Proposition syn.1 by **??**. □

Moreover, the value of a term, and whether or not a structure satisfies a formula, only depend on the values of its subterms.

fol:syn:ext: prop:ext-terms

Proposition syn.3. *Let $\mathfrak{M}$ be a structure, t and t' terms, and s a variable assignment. Then $\mathrm{Val}^{\mathfrak{M}}_s(t[t'/x]) = \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]}(t)$.*

Proof. By induction on t.

1. If t is a constant, say, $t \equiv c$, then $t[t'/x] = c$, and $\mathrm{Val}^{\mathfrak{M}}_s(c) = c^{\mathfrak{M}} = \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]}(c)$.

2. If t is a variable other than x, say, $t \equiv y$, then $t[t'/x] = y$, and $\mathrm{Val}^{\mathfrak{M}}_s(y) = \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]}(y)$ since $s \sim_x s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]$.

3. If $t \equiv x$, then $t[t'/x] = t'$. But $\mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]}(x) = \mathrm{Val}^{\mathfrak{M}}_s(t')$ by definition of $s[\mathrm{Val}^{\mathfrak{M}}_s(t')/x]$.

4. If $t \equiv f(t_1, \dots, t_n)$ then we have:

$$\begin{aligned}
&\mathrm{Val}^{\mathfrak{M}}_{s}(t[t'/x]) = \\
&\qquad = \mathrm{Val}^{\mathfrak{M}}_{s}(f(t_1[t'/x], \dots, t_n[t'/x])) \\
&\qquad\qquad \text{by definition of } t[t'/x] \\
&\qquad = f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s}(t_1[t'/x]), \dots, \mathrm{Val}^{\mathfrak{M}}_{s}(t_n[t'/x])) \\
&\qquad\qquad \text{by definition of } \mathrm{Val}^{\mathfrak{M}}_{s}(f(\dots)) \\
&\qquad = f^{\mathfrak{M}}(\mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_{s}(t')/x]}(t_1), \dots, \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_{s}(t')/x]}(t_n)) \\
&\qquad\qquad \text{by induction hypothesis} \\
&\qquad = \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_{s}(t')/x]}(t) \text{ by definition of } \mathrm{Val}^{\mathfrak{M}}_{s[\mathrm{Val}^{\mathfrak{M}}_{s}(t')/x]}(f(\dots)) \qquad \square
\end{aligned}$$

Proposition syn.4. *Let $\mathfrak{M}$ be a structure, φ a formula, t' a term, and s a variable assignment. Then $\mathfrak{M}, s \vDash \varphi[t'/x]$ iff $\mathfrak{M}, s[\mathrm{Val}^{\mathfrak{M}}_{s}(t')/x] \vDash \varphi$.* *fol:syn:ext:prop:ext-formulas*

Proof. Exercise. □

Problem syn.2. Prove Proposition syn.4

explanation The point of Propositions syn.3 and syn.4 is the following. Suppose we have a term t or a formula φ and some term t', and we want to know the value of $t[t'/x]$ or whether or not $\varphi[t'/x]$ is satisfied in a structure $\mathfrak{M}$ relative to a variable assignment s. Then we can either perform the substitution first and then consider the value or satisfaction relative to $\mathfrak{M}$ and s, or we can first determine the value $m = \mathrm{Val}^{\mathfrak{M}}_{s}(t')$ of t' in $\mathfrak{M}$ relative to s, change the variable assignment to $s[m/x]$ and then consider the value of t in $\mathfrak{M}$ and $s[m/x]$, or whether $\mathfrak{M}, s[m/x] \vDash \varphi$. Propositions syn.3 and syn.4 guarantee that the answer will be the same, whichever way we do it.

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Bibliography