

syn.1 Covered Structures for First-order Languages

fol:syn:cov:
sec

Recall that a term is *closed* if it contains no **variables**.

explanation

Definition syn.1 (Value of closed terms). If t is a closed term of the language \mathcal{L} and \mathfrak{M} is a **structure** for \mathcal{L} , the **value** $\text{Val}^{\mathfrak{M}}(t)$ is defined as follows:

1. If t is just the **constant symbol** c , then $\text{Val}^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$.
2. If t is of the form $f(t_1, \dots, t_n)$, then

$$\text{Val}^{\mathfrak{M}}(t) = f^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(t_1), \dots, \text{Val}^{\mathfrak{M}}(t_n)).$$

Definition syn.2 (Covered structure). A **structure** is *covered* if every element of the domain is the **value** of some closed term.

Example syn.3. Let \mathcal{L} be the language with **constant symbols** *zero*, *one*, *two*, \dots , the binary **predicate symbol** $<$, and the binary **function symbols** $+$ and \times . Then a **structure** \mathfrak{M} for \mathcal{L} is the one with domain $|\mathfrak{M}| = \{0, 1, 2, \dots\}$ and assignments $zero^{\mathfrak{M}} = 0$, $one^{\mathfrak{M}} = 1$, $two^{\mathfrak{M}} = 2$, and so forth. For the binary relation symbol $<$, the set $<^{\mathfrak{M}}$ is the set of all pairs $\langle c_1, c_2 \rangle \in |\mathfrak{M}|^2$ such that c_1 is less than c_2 : for example, $\langle 1, 3 \rangle \in <^{\mathfrak{M}}$ but $\langle 2, 2 \rangle \notin <^{\mathfrak{M}}$. For the binary **function symbol** $+$, define $+^{\mathfrak{M}}$ in the usual way—for example, $+^{\mathfrak{M}}(2, 3)$ maps to 5, and similarly for the binary **function symbol** \times . Hence, the **value** of *four* is just 4, and the **value** of $\times(\textit{two}, +(\textit{three}, \textit{zero}))$ (or in infix notation, $\textit{two} \times (\textit{three} + \textit{zero})$) is

$$\begin{aligned} \text{Val}^{\mathfrak{M}}(\times(\textit{two}, +(\textit{three}, \textit{zero}))) &= \\ &= \times^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\textit{two}), \text{Val}^{\mathfrak{M}}(\textit{two}, +(\textit{three}, \textit{zero}))) \\ &= \times^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\textit{two}), +^{\mathfrak{M}}(\text{Val}^{\mathfrak{M}}(\textit{three}), \text{Val}^{\mathfrak{M}}(\textit{zero}))) \\ &= \times^{\mathfrak{M}}(\textit{two}^{\mathfrak{M}}, +^{\mathfrak{M}}(\textit{three}^{\mathfrak{M}}, \textit{zero}^{\mathfrak{M}})) \\ &= \times^{\mathfrak{M}}(2, +^{\mathfrak{M}}(3, 0)) \\ &= \times^{\mathfrak{M}}(2, 3) \\ &= 6 \end{aligned}$$

Problem syn.1. Is \mathfrak{N} , the standard model of arithmetic, covered? Explain.

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Bibliography