

## seq.1 Soundness with Identity predicate

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**Proposition seq.1.** **LK** with initial sequents and rules for identity is sound.

*Proof.* Initial sequents of the form  $\Rightarrow t = t$  are valid, since for every **structure**  $\mathfrak{M}$ ,  $\mathfrak{M} \models t = t$ . (Note that we assume the term  $t$  to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a **derivation** is  $=$ . Then the premise is  $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$  and the conclusion is  $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$ . Consider a **structure**  $\mathfrak{M}$ . We need to show that the conclusion is valid, i.e., if  $\mathfrak{M} \models t_1 = t_2$  and  $\mathfrak{M} \models \Gamma$ , then either  $\mathfrak{M} \models \chi$  for some  $\chi \in \Delta$  or  $\mathfrak{M} \models \varphi(t_2)$ .

By induction hypothesis, the premise is valid. This means that if  $\mathfrak{M} \models t_1 = t_2$  and  $\mathfrak{M} \models \Gamma$  either (a) for some  $\chi \in \Delta$ ,  $\mathfrak{M} \models \chi$  or (b)  $\mathfrak{M} \models \varphi(t_1)$ . In case (a) we are done. Consider case (b). Let  $s$  be a variable assignment with  $s(x) = \text{Val}^{\mathfrak{M}}(t_1)$ . By ??,  $\mathfrak{M}, s \models \varphi(t_1)$ . Since  $s \sim_x s$ , by ??,  $\mathfrak{M}, s \models \varphi(x)$ . since  $\mathfrak{M} \models t_1 = t_2$ , we have  $\text{Val}^{\mathfrak{M}}(t_1) = \text{Val}^{\mathfrak{M}}(t_2)$ , and hence  $s(x) = \text{Val}^{\mathfrak{M}}(t_2)$ . By applying ?? again, we also have  $\mathfrak{M}, s \models \varphi(t_2)$ . By ??,  $\mathfrak{M} \models \varphi(t_2)$ .  $\square$

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## Bibliography