Chapter udf

The Sequent Calculus

This chapter presents Gentzen's standard sequent calculus LK for classical first-order logic. It could use more examples and exercises. To include or exclude material relevant to the sequent calculus as a proof system, use the "prfLK" tag.

seq.1 Rules and Derivations

fol:seq:rul:sec For the following, let $\Gamma, \Delta, \Pi, \Lambda$ represent finite sequences of sentences.

Definition seq.1 (Sequent). A *sequent* is an expression of the form

$$\Gamma \Rightarrow \Delta$$

where Γ and Δ are finite (possibly empty) sequences of sentences of the language $\mathcal{L}$. Γ is called the *antecedent*, while Δ is the *succedent*.

The intuitive idea behind a sequent is: if all of the sentences in the antecedent hold, then at least one of the sentences in the succedent holds. That is, if $\Gamma = \langle \varphi_1, \ldots, \varphi_m \rangle$ and $\Delta = \langle \psi_1, \ldots, \psi_n \rangle$, then $\Gamma \Rightarrow \Delta$ holds iff explanation

$$(\varphi_1 \land \cdots \land \varphi_m) \to (\psi_1 \lor \cdots \lor \psi_n)$$

holds. There are two special cases: where Γ is empty and when Δ is empty. When Γ is empty, i.e., $m = 0$, $\Rightarrow \Delta$ holds iff $\psi_1 \lor \cdots \lor \psi_n$ holds. When Δ is empty, i.e., $n = 0$, $\Gamma \Rightarrow$ holds iff $\lnot(\varphi_1 \land \cdots \land \varphi_m)$ does. We say a sequent is valid iff the corresponding sentence is valid.

If Γ is a sequence of sentences, we write Γ, φ for the result of appending φ to the right end of Γ (and φ, Γ for the result of appending φ to the left end of Γ). If Δ is a sequence of sentences also, then Γ, Δ is the concatenation of the two sequences.

Definition seq.2 (Initial Sequent). An *initial sequent* is a sequent of one of the following forms:

1. $\varphi \Rightarrow \varphi$
2. $\Rightarrow \top$
3. $\bot \Rightarrow$

for any sentence φ in the language.

Derivations in the sequent calculus are certain trees of sequents, where the topmost sequents are initial sequents, and if a sequent stands below one or two other sequents, it must follow correctly by a rule of inference. The rules for **LK** are divided into two main types: *logical* rules and *structural* rules. The logical rules are named for the main operator of the sentence containing φ and/or ψ in the lower sequent. Each one comes in two versions, one for inferring a sequent with the sentence containing the logical operator on the left, and one with the sentence on the right.

seq.2 Propositional Rules

Rules for ¬ fol:seq:prl:sec

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\ \neg\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ \neg\mathrm{R}$$

Rules for ∧

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta}\ \land\mathrm{L} \qquad \frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \land \psi, \Gamma \Rightarrow \Delta}\ \land\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi}\ \land\mathrm{R}$$

Rules for ∨

$$\frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \lor \psi, \Gamma \Rightarrow \Delta}\ \lor\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}\ \lor\mathrm{R} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}\ \lor\mathrm{R}$$

Rules for $\to$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Pi \Rightarrow \Lambda}{\varphi \to \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \to\mathrm{L} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ \to\mathrm{R}$$

seq.3 Quantifier Rules

fol:seq:qrl:sec

Rules for $\forall$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \forall\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ \forall\mathrm{R}$$

In $\forall$L, t is a closed term (i.e., one without variables). In $\forall$R, a is a constant symbol which must not occur anywhere in the lower sequent of the $\forall$R rule. We call a the *eigenvariable* of the $\forall$R inference.[1]

Rules for $\exists$

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \exists\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x\, \varphi(x)}\ \exists\mathrm{R}$$

Again, t is a closed term, and a is a constant symbol which does not occur in the lower sequent of the $\exists$L rule. We call a the *eigenvariable* of the $\exists$L inference.

The condition that an eigenvariable not occur in the lower sequent of the $\forall$R or $\exists$L inference is called the *eigenvariable condition*.

Recall the convention that when φ is a formula with the variable x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists$R rule as: explanation

$$\frac{\Gamma \Rightarrow \Delta, \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x\, \varphi}\ \exists\mathrm{R}$$

Note that t may already occur in φ, e.g., φ might be $P(t, x)$. Thus, inferring $\Gamma \Rightarrow \Delta, \exists x\, P(t, x)$ from $\Gamma \Rightarrow \Delta, P(t, t)$ is a correct application of $\exists$R—you may "replace" one or more, and not necessarily all, occurrences of t in the premise by the bound variable x. However, the eigenvariable conditions in $\forall$R

[1] We use the term "eigenvariable" even though a in the above rule is a constant symbol. This has historical reasons.

and $\exists$L require that the constant symbol a does not occur in φ. So, you cannot correctly infer $\Gamma \Rightarrow \Delta, \forall x\, P(a, x)$ from $\Gamma \Rightarrow \Delta, P(a, a)$ using $\forall$R.

explanation

In $\exists$R and $\forall$L there are no restrictions on the term t. On the other hand, in the $\exists$L and $\forall$R rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere outside of $\varphi(a)$ in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed:

$$\dfrac{\dfrac{\varphi(a) \Rightarrow \varphi(a)}{\exists x\, \varphi(x) \Rightarrow \varphi(a)}\ {*\exists\mathrm{L}}}{\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)}\ \forall\mathrm{R} \qquad \dfrac{\dfrac{\varphi(a) \Rightarrow \varphi(a)}{\varphi(a) \Rightarrow \forall x\, \varphi(x)}\ {*\forall\mathrm{R}}}{\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)}\ \exists\mathrm{L}$$

However, $\exists x\, \varphi(x) \Rightarrow \forall x\, \varphi(x)$ is not valid.

seq.4 Structural Rules

We also need a few rules that allow us to rearrange sentences in the left and right side of a sequent. Since the logical rules require that the sentences in the premise which the rule acts upon stand either to the far left or to the far right, we need an "exchange" rule that allows us to move sentences to the right position. It's also important sometimes to be able to combine two identical sentences into one, and to add a sentence on either side. fol:seq:srl:sec

Weakening

$$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \mathrm{WL} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \mathrm{WR}$$

Contraction

$$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \mathrm{CL} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ \mathrm{CR}$$

Exchange

$$\dfrac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}\ \mathrm{XL} \qquad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}\ \mathrm{XR}$$

A series of weakening, contraction, and exchange inferences will often be indicated by double inference lines.

The following rule, called "cut," is not strictly speaking necessary, but makes it a lot easier to reuse and combine derivations.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

seq.5 Derivations

fol:seq:der:sec We've said what an initial sequent looks like, and we've given the rules of inference. Derivations in the sequent calculus are inductively generated from these: each derivation either is an initial sequent on its own, or consists of one or two derivations followed by an inference. explanation

Definition seq.3 (LK derivation). An **LK**-*derivation* of a sequent S is a finite tree of sequents satisfying the following conditions:

1. The topmost sequents of the tree are initial sequents.
2. The bottommost sequent of the tree is S.
3. Every sequent in the tree except S is a premise of a correct application of an inference rule whose conclusion stands directly below that sequent in the tree.

We then say that S is the *end-sequent* of the derivation and that S is *derivable in* **LK** (or **LK**-derivable).

Example seq.4. Every initial sequent, e.g., $\chi \Rightarrow \chi$ is a derivation. We can obtain a new derivation from this by applying, say, the WL rule,

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{WL}$$

The rule, however, is meant to be general: we can replace the φ in the rule with any sentence, e.g., also with θ. If the premise matches our initial sequent $\chi \Rightarrow \chi$, that means that both Γ and Δ are just χ, and the conclusion would then be $\theta, \chi \Rightarrow \chi$. So, the following is a derivation:

$$\frac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}$$

We can now apply another rule, say XL, which allows us to switch two sentences on the left. So, the following is also a correct derivation:

$$\cfrac{\cfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}}{\chi, \theta \Rightarrow \chi}\ \text{XL}$$

In this application of the rule, which was given as

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta,}\ \text{XL}$$

both Γ and Π were empty, Δ is χ, and the roles of φ and ψ are played by θ and χ, respectively. In much the same way, we also see that

$$\frac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\ \text{WL}$$

is a derivation. Now we can take these two derivations, and combine them using $\wedge$R. That rule was

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge\text{R}$$

In our case, the premises must match the last sequents of the derivations ending in the premises. That means that Γ is χ, θ, Δ is empty, φ is χ and ψ is θ. So the conclusion, if the inference should be correct, is $\chi, \theta \Rightarrow \chi \wedge \theta$.

$$\frac{\dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}}{\chi, \theta \Rightarrow \chi}\ \text{XL} \qquad \dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\ \text{WL}}{\chi, \theta \Rightarrow \chi \wedge \theta}\ \wedge\text{R}$$

Of course, we can also reverse the premises, then φ would be θ and ψ would be χ.

$$\frac{\dfrac{\theta \Rightarrow \theta}{\chi, \theta \Rightarrow \theta}\ \text{WL} \qquad \dfrac{\dfrac{\chi \Rightarrow \chi}{\theta, \chi \Rightarrow \chi}\ \text{WL}}{\chi, \theta \Rightarrow \chi}\ \text{XL}}{\chi, \theta \Rightarrow \theta \wedge \chi}\ \wedge\text{R}$$

seq.6 Examples of Derivations

fol:seq:pro:sec

Example seq.5. Give an **LK**-derivation for the sequent $\varphi \wedge \psi \Rightarrow \varphi$.

We begin by writing the desired end-sequent at the bottom of the derivation.

$$\frac{}{\varphi \wedge \psi \Rightarrow \varphi}$$

Next, we need to figure out what kind of inference could have a lower sequent of this form. This could be a structural rule, but it is a good idea to start by looking for a logical rule. The only logical connective occurring in the lower sequent is $\wedge$, so we're looking for an $\wedge$ rule, and since the $\wedge$ symbol occurs in the antecedent, we're looking at the $\wedge$L rule.

$$\frac{}{\varphi \wedge \psi \Rightarrow \varphi}\ \wedge\text{L}$$

There are two options for what could have been the upper sequent of the $\land$L inference: we could have an upper sequent of $\varphi \Rightarrow \varphi$, or of $\psi \Rightarrow \varphi$. Clearly, $\varphi \Rightarrow \varphi$ is an initial sequent (which is a good thing), while $\psi \Rightarrow \varphi$ is not derivable in general. We fill in the upper sequent:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\,\land\text{L}$$

We now have a correct **LK**-derivation of the sequent $\varphi \land \psi \Rightarrow \varphi$.

Example seq.6. Give an **LK**-derivation for the sequent $\neg\varphi \lor \psi \Rightarrow \varphi \to \psi$.

Begin by writing the desired end-sequent at the bottom of the derivation.

$$\dfrac{}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}$$

To find a logical rule that could give us this end-sequent, we look at the logical connectives in the end-sequent: $\neg$, $\lor$, and $\to$. We only care at the moment about $\lor$ and $\to$ because they are main operators of sentences in the end-sequent, while $\neg$ is inside the scope of another connective, so we will take care of it later. Our options for logical rules for the final inference are therefore the $\lor$L rule and the $\to$R rule. We could pick either rule, really, but let's pick the $\to$R rule (if for no reason other than it allows us to put off splitting into two branches). According to the form of $\to$R inferences which can yield the lower sequent, this must look like:

$$\dfrac{\dfrac{}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,\to\text{R}$$

If we move $\neg\varphi \lor \psi$ to the outside of the antecedent, we can apply the $\lor$L rule. According to the schema, this must split into two upper sequents as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi, \varphi \Rightarrow \psi} \qquad \dfrac{}{\psi, \varphi \Rightarrow \psi}}{\neg\varphi \lor \psi, \varphi \Rightarrow \psi}\,\lor\text{L}}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}\,\text{XR}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,\to\text{R}$$

Remember that we are trying to wind our way up to initial sequents; we seem to be pretty close! The right branch is just one weakening and one exchange away from an initial sequent and then it is done:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi, \varphi \Rightarrow \psi} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\,\text{WL}}{\psi, \varphi \Rightarrow \psi}\,\text{XL}}{\neg\varphi \lor \psi, \varphi \Rightarrow \psi}\,\lor\text{L}}{\varphi, \neg\varphi \lor \psi \Rightarrow \psi}\,\text{XR}}{\neg\varphi \lor \psi \Rightarrow \varphi \to \psi}\,\to\text{R}$$

Now looking at the left branch, the only logical connective in any sentence is the $\neg$ symbol in the antecedent sentences, so we're looking at an instance of the $\neg$L rule.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi \Rightarrow \psi, \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow \psi$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{WL}
\UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
\RightLabel{XL}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\RightLabel{$\vee$L}
\BinaryInfC{$\neg\varphi \vee \psi, \varphi \Rightarrow \psi$}
\RightLabel{XR}
\UnaryInfC{$\varphi, \neg\varphi \vee \psi \Rightarrow \psi$}
\RightLabel{$\to$R}
\UnaryInfC{$\neg\varphi \vee \psi \Rightarrow \varphi \to \psi$}
\end{prooftree}
$$

Similarly to how we finished off the right branch, we are just one weakening and one exchange away from finishing off this left branch as well.

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{WR}
\UnaryInfC{$\varphi \Rightarrow \varphi, \psi$}
\RightLabel{XR}
\UnaryInfC{$\varphi \Rightarrow \psi, \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow \psi$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{WL}
\UnaryInfC{$\varphi, \psi \Rightarrow \psi$}
\RightLabel{XL}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\RightLabel{$\vee$L}
\BinaryInfC{$\neg\varphi \vee \psi, \varphi \Rightarrow \psi$}
\RightLabel{XR}
\UnaryInfC{$\varphi, \neg\varphi \vee \psi \Rightarrow \psi$}
\RightLabel{$\to$R}
\UnaryInfC{$\neg\varphi \vee \psi \Rightarrow \varphi \to \psi$}
\end{prooftree}
$$

Example seq.7. Give an **LK**-derivation of the sequent $\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$

Using the techniques from above, we start by writing the desired end-sequent at the bottom.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

The available main connectives of sentences in the end-sequent are the $\vee$ symbol and the $\neg$ symbol. It would work to apply either the $\vee$L or the $\neg$R rule here, but we start with the $\neg$R rule because it avoids splitting up into two branches for a moment:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

Now we have a choice of whether to look at the $\wedge$L or the $\vee$L rule. Let's see what happens when we apply the $\wedge$L rule: we have a choice to start with either the sequent $\varphi, \neg\varphi \vee \psi \Rightarrow$ or the sequent $\psi, \neg\varphi \vee \psi \Rightarrow$. Since the derivation is symmetric with regards to φ and ψ, let's go with the former:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\varphi, \neg\varphi \vee \neg\psi \Rightarrow$}
\RightLabel{$\wedge$L}
\UnaryInfC{$\varphi \wedge \psi, \neg\varphi \vee \neg\psi \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\neg\varphi \vee \neg\psi \Rightarrow \neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

Continuing to fill in the derivation, we see that we run into a problem:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\,\neg\text{L} \qquad \dfrac{\dfrac{}{\varphi \Rightarrow \psi}\,?}{\neg\psi, \varphi \Rightarrow}\,\neg\text{L}}{\neg\varphi \lor \neg\psi, \varphi \Rightarrow}\,\lor\text{L}}{\varphi, \neg\varphi \lor \neg\psi \Rightarrow}\,\text{XL}}{\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow}\,\land\text{L}}{\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)}\,\neg\text{R}
$$

The top of the right branch cannot be reduced any further, and it cannot be brought by way of structural inferences to an initial sequent, so this is not the right path to take. So clearly, it was a mistake to apply the $\land$L rule above. Going back to what we had before and carrying out the $\lor$L rule instead, we get

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\varphi, \varphi \land \psi \Rightarrow} \qquad \dfrac{}{\neg\psi, \varphi \land \psi \Rightarrow}}{\neg\varphi \lor \neg\psi, \varphi \land \psi \Rightarrow}\,\lor\text{L}}{\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow}\,\text{XL}}{\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)}\,\neg\text{R}
$$

Completing each branch as we've done before, we get

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\,\land\text{L}}{\neg\varphi, \varphi \land \psi \Rightarrow}\,\neg\text{L} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi \land \psi \Rightarrow \psi}\,\land\text{L}}{\neg\psi, \varphi \land \psi \Rightarrow}\,\neg\text{L}}{\neg\varphi \lor \neg\psi, \varphi \land \psi \Rightarrow}\,\lor\text{L}}{\varphi \land \psi, \neg\varphi \lor \neg\psi \Rightarrow}\,\text{XL}}{\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)}\,\neg\text{R}
$$

(We could have carried out the $\land$ rules lower than the $\neg$ rules in these steps and still obtained a correct derivation).

Example seq.8. So far we haven't used the contraction rule, but it is sometimes required. Here's an example where that happens. Suppose we want to prove $\Rightarrow \varphi \lor \neg\varphi$. Applying $\lor$R backwards would give us one of these two derivations:

$$
\dfrac{\dfrac{}{\Rightarrow \varphi}}{\Rightarrow \varphi \lor \neg\varphi}\,\lor\text{R} \qquad\qquad \dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow}}{\Rightarrow \neg\varphi}\,\neg\text{R}}{\Rightarrow \varphi \lor \neg\varphi}\,\lor\text{R}
$$

Neither of these of course ends in an initial sequent. The trick is to realize that the contraction rule allows us to combine two copies of a sentence into one—and when we're searching for a proof, i.e., going from bottom to top, we can keep a copy of $\varphi \lor \neg\varphi$ in the premise, e.g.,

$$
\dfrac{\dfrac{\dfrac{}{\Rightarrow \varphi \lor \neg\varphi, \varphi}}{\Rightarrow \varphi \lor \neg\varphi, \varphi \lor \neg\varphi}\,\lor\text{R}}{\Rightarrow \varphi \lor \neg\varphi}\,\text{CR}
$$

Now we can apply $\lor$R a second time, and also get $\neg\varphi$, which leads to a complete derivation.

$$
\begin{array}{ll}
\varphi \Rightarrow \varphi & \\ \hline
\Rightarrow \varphi, \neg\varphi & \neg\text{R} \\ \hline
\Rightarrow \varphi, \varphi \lor \neg\varphi & \lor\text{R} \\ \hline
\Rightarrow \varphi \lor \neg\varphi, \varphi & \text{XR} \\ \hline
\Rightarrow \varphi \lor \neg\varphi, \varphi \lor \neg\varphi & \lor\text{R} \\ \hline
\Rightarrow \varphi \lor \neg\varphi & \text{CR}
\end{array}
$$

Problem seq.1. Give derivations of the following sequents:

1. $\varphi \land (\psi \land \chi) \Rightarrow (\varphi \land \psi) \land \chi$.
2. $\varphi \lor (\psi \lor \chi) \Rightarrow (\varphi \lor \psi) \lor \chi$.
3. $\varphi \to (\psi \to \chi) \Rightarrow \psi \to (\varphi \to \chi)$.
4. $\varphi \Rightarrow \neg\neg\varphi$.

Problem seq.2. Give derivations of the following sequents:

1. $(\varphi \lor \psi) \to \chi \Rightarrow \varphi \to \chi$.
2. $(\varphi \to \chi) \land (\psi \to \chi) \Rightarrow (\varphi \lor \psi) \to \chi$.
3. $\Rightarrow \neg(\varphi \land \neg\varphi)$.
4. $\psi \to \varphi \Rightarrow \neg\varphi \to \neg\psi$.
5. $\Rightarrow (\varphi \to \neg\varphi) \to \neg\varphi$.
6. $\Rightarrow \neg(\varphi \to \psi) \to \neg\psi$.
7. $\varphi \to \chi \Rightarrow \neg(\varphi \land \neg\chi)$.
8. $\varphi \land \neg\chi \Rightarrow \neg(\varphi \to \chi)$.
9. $\varphi \lor \psi, \neg\psi \Rightarrow \varphi$.
10. $\neg\varphi \lor \neg\psi \Rightarrow \neg(\varphi \land \psi)$.
11. $\Rightarrow (\neg\varphi \land \neg\psi) \to \neg(\varphi \lor \psi)$.
12. $\Rightarrow \neg(\varphi \lor \psi) \to (\neg\varphi \land \neg\psi)$.

Problem seq.3. Give derivations of the following sequents:

1. $\neg(\varphi \to \psi) \Rightarrow \varphi$.
2. $\neg(\varphi \land \psi) \Rightarrow \neg\varphi \lor \neg\psi$.
3. $\varphi \to \psi \Rightarrow \neg\varphi \lor \psi$.

4. $\Rightarrow \neg\neg\varphi \to \varphi$.
5. $\varphi \to \psi, \neg\varphi \to \psi \Rightarrow \psi$.
6. $(\varphi \wedge \psi) \to \chi \Rightarrow (\varphi \to \chi) \vee (\psi \to \chi)$.
7. $(\varphi \to \psi) \to \varphi \Rightarrow \varphi$.
8. $\Rightarrow (\varphi \to \psi) \vee (\psi \to \chi)$.

(These all require the CR rule.)

seq.7 Derivations with Quantifiers

fol:seq:prq:sec

Example seq.9. Give an **LK**-derivation of the sequent $\exists x\, \neg\varphi(x) \Rightarrow \neg\forall x\, \varphi(x)$.

When dealing with quantifiers, we have to make sure not to violate the eigenvariable condition, and sometimes this requires us to play around with the order of carrying out certain inferences. In general, it helps to try and take care of rules subject to the eigenvariable condition first (they will be lower down in the finished proof). Also, it is a good idea to try and look ahead and try to guess what the initial sequent might look like. In our case, it will have to be something like $\varphi(a) \Rightarrow \varphi(a)$. That means that when we are "reversing" the quantifier rules, we will have to pick the same term—what we will call a—for both the $\forall$ and the $\exists$ rule. If we picked different terms for each rule, we would end up with something like $\varphi(a) \Rightarrow \varphi(b)$, which, of course, is not derivable.

Starting as usual, we write

$$\dfrac{}{\exists x\, \neg\varphi(x) \;\Rightarrow\; \neg\forall x\, \varphi(x)}$$

We could either carry out the $\exists$L rule or the $\neg$R rule. Since the $\exists$L rule is subject to the eigenvariable condition, it's a good idea to take care of it sooner rather than later, so we'll do that one first.

$$\dfrac{\dfrac{}{\neg\varphi(a) \;\Rightarrow\; \neg\forall x\, \varphi(x)}}{\exists x\, \neg\varphi(x) \;\Rightarrow\; \neg\forall x\, \varphi(x)}\;\exists\mathrm{L}$$

Applying the $\neg$L and $\neg$R rules backwards, we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x\, \varphi(x) \;\Rightarrow\; \varphi(a)}}{\neg\varphi(a), \forall x\, \varphi(x) \;\Rightarrow}\;\neg\mathrm{L}}{\forall x\, \varphi(x), \neg\varphi(a) \;\Rightarrow}\;\mathrm{XL}}{\neg\varphi(a) \;\Rightarrow\; \neg\forall x \varphi(x)}\;\neg\mathrm{R}}{\exists x \neg\varphi(x) \;\Rightarrow\; \neg\forall x \varphi(x)}\;\exists\mathrm{L}$$

At this point, our only option is to carry out the $\forall$L rule. Since this rule is not subject to the eigenvariable restriction, we're in the clear. Remember, we want to try and obtain an initial sequent (of the form $\varphi(a) \Rightarrow \varphi(a)$), so we should choose a as our argument for φ when we apply the rule.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi(a) \Rightarrow \varphi(a)}{\forall x\,\varphi(x) \Rightarrow \varphi(a)}\,\forall\mathrm{L}}{\neg\varphi(a), \forall x\,\varphi(x) \Rightarrow}\,\neg\mathrm{L}}{\forall x\,\varphi(x), \neg\varphi(a) \Rightarrow}\,\mathrm{XL}}{\neg\varphi(a) \Rightarrow \neg\forall x\,\varphi(x)}\,\neg\mathrm{R}}{\exists x\,\neg\varphi(x) \Rightarrow \neg\forall x\,\varphi(x)}\,\exists\mathrm{L}
$$

It is important, especially when dealing with quantifiers, to double check at this point that the eigenvariable condition has not been violated. Since the only rule we applied that is subject to the eigenvariable condition was $\exists$L, and the eigenvariable a does not occur in its lower sequent (the end-sequent), this is a correct derivation.

Problem seq.4. Give derivations of the following sequents:

1. $\Rightarrow (\forall x\,\varphi(x) \land \forall y\,\psi(y)) \to \forall z\,(\varphi(z) \land \psi(z))$.
2. $\Rightarrow (\exists x\,\varphi(x) \lor \exists y\,\psi(y)) \to \exists z\,(\varphi(z) \lor \psi(z))$.
3. $\forall x\,(\varphi(x) \to \psi) \Rightarrow \exists y\,\varphi(y) \to \psi$.
4. $\forall x\,\neg\varphi(x) \Rightarrow \neg\exists x\,\varphi(x)$.
5. $\Rightarrow \neg\exists x\,\varphi(x) \to \forall x\,\neg\varphi(x)$.
6. $\Rightarrow \neg\exists x\,\forall y\,((\varphi(x,y) \to \neg\varphi(y,y)) \land (\neg\varphi(y,y) \to \varphi(x,y)))$.

Problem seq.5. Give derivations of the following sequents:

1. $\Rightarrow \neg\forall x\,\varphi(x) \to \exists x\,\neg\varphi(x)$.
2. $(\forall x\,\varphi(x) \to \psi) \Rightarrow \exists y\,(\varphi(y) \to \psi)$.
3. $\Rightarrow \exists x\,(\varphi(x) \to \forall y\,\varphi(y))$.

(These all require the CR rule.)

> This section collects the definitions of the provability relation and consistency for natural deduction.

seq.8 Proof-Theoretic Notions

explanation Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions.* These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem.* fol:seq:ptn:sec

Definition seq.10 (Theorems). A sentence φ is a *theorem* if there is a derivation in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition seq.11 (Derivability). A sentence φ is *derivable from* a set of sentences Γ, $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ_0' of the sentences in Γ_0 such that **LK** derives $\Gamma_0' \Rightarrow \varphi$. If φ is not derivable from Γ we write $\Gamma \nvdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of sentences in Γ_0' does not matter: if a sequent $\Gamma_0' \Rightarrow \varphi$ is derivable, then so is $\Gamma_0'' \Rightarrow \varphi$ for any Γ_0'' that contains the same sentences as Γ_0'. For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma_0' = \langle \psi, \psi, \chi \rangle$ and $\Gamma_0'' = \langle \chi, \chi, \psi \rangle$ are sequences containing just the sentences in Γ_0. If a sequent containing one is derivable, so is the other, e.g.:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \psi, \psi, \chi \Rightarrow \varphi\end{matrix}}{\psi, \chi \Rightarrow \varphi}\,\text{CL}}{\chi, \psi \Rightarrow \varphi}\,\text{XL}}{\chi, \chi, \psi \Rightarrow \varphi}\,\text{WL}$$

From now on we'll say that if Γ_0 is a finite set of sentences then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of sentences in Γ_0 and tacitly include contractions, exchanges, and weakenings if necessary.

Definition seq.12 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. If Γ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not derive $\Gamma_0 \Rightarrow$, we say it is *consistent*.

fol:seq:ptn: prop:reflexivity

Proposition seq.13 (Reflexivity). *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The initial sequent $\varphi \Rightarrow \varphi$ is derivable, and $\{\varphi\} \subseteq \Gamma$. □

fol:seq:ptn: prop:monotonicity

Proposition seq.14 (Monotonicity). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Suppose $\Gamma \vdash \varphi$, i.e., there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is derivable. Since $\Gamma \subseteq \Delta$, then Γ_0 is also a finite subset of Δ. The derivation of $\Gamma_0 \Rightarrow \varphi$ thus also shows $\Delta \vdash \varphi$. □

fol:seq:ptn: prop:transitivity

Proposition seq.15 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. If $\Gamma \vdash \varphi$, there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation π_0 of $\Gamma_0 \Rightarrow \varphi$. If $\{\varphi\} \cup \Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a derivation π_1 of $\varphi, \Delta_0 \Rightarrow \psi$. Consider the following derivation:

$$\dfrac{\dfrac{\vdots\ \pi_0}{\Gamma_0 \Rightarrow \varphi} \qquad \dfrac{\vdots\ \pi_1}{\varphi, \Delta_0 \Rightarrow \psi}}{\Gamma_0, \Delta_0 \Rightarrow \psi}\ \text{Cut}$$

Since $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$, this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

Proposition seq.16. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ.* fol:seq:ptn:prop:incons

Proof. Exercise. □

Problem seq.6. Prove Proposition seq.16

Proposition seq.17 (Compactness). fol:seq:ptn:prop:proves-compact

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a derivation. Consequently, $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. But then Γ_0 is a finite subset of Γ that is inconsistent. □

seq.9 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem. fol:seq:prv:sec

Proposition seq.18. *If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then Γ is inconsistent.* fol:seq:prv:prop:provability-contr

Proof. There are finite Γ_0 and $\Gamma_1 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow$. Let the **LK**-derivation of $\Gamma_0 \Rightarrow \varphi$ be π_0 and the **LK**-derivation of $\Gamma_1, \varphi \Rightarrow$ be π_1. We can then derive

$$\dfrac{\dfrac{\vdots\ \pi_0}{\Gamma_0 \Rightarrow \varphi} \qquad \dfrac{\vdots\ \pi_1}{\varphi, \Gamma_1 \Rightarrow}}{\Gamma_0, \Gamma_1 \Rightarrow}\ \text{Cut}$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence Γ is inconsistent. □

fol:seq:prv: prop:prov-incons **Proposition seq.19.** *$\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is inconsistent.*

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation π_0 of $\Gamma \Rightarrow \varphi$. By adding a $\neg$L rule, we obtain a derivation of $\neg\varphi, \Gamma \Rightarrow$, i.e., $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg\varphi\}$ is inconsistent, there is a derivation π_1 of $\neg\varphi, \Gamma \Rightarrow$. The following is a derivation of $\Gamma \Rightarrow \varphi$:

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Rightarrow \varphi, \neg\varphi$}
\AxiomC{$\vdots\ \pi_1$}
\noLine
\UnaryInfC{$\neg\varphi, \Gamma \Rightarrow$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma \Rightarrow \varphi$}
\end{prooftree}
$$

□

Problem seq.7. Prove that $\Gamma \vdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

fol:seq:prv: prop:explicit-inc **Proposition seq.20.** *If $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$, then Γ is inconsistent.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\neg\varphi \in \Gamma$. Then there is a derivation π of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg\varphi, \Gamma_0 \Rightarrow$ is also derivable:

$$
\begin{prooftree}
\AxiomC{$\vdots\ \pi$}
\noLine
\UnaryInfC{$\Gamma_0 \Rightarrow \varphi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\varphi, \varphi \Rightarrow$}
\RightLabel{XL}
\UnaryInfC{$\varphi, \neg\varphi \Rightarrow$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_0, \neg\varphi \Rightarrow$}
\end{prooftree}
$$

Since $\neg\varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that Γ is inconsistent. □

fol:seq:prv: prop:provability-exhaustive **Proposition seq.21.** *If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are both inconsistent, then Γ is inconsistent.*

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and **LK**-derivations π_0 and π_1 of $\varphi, \Gamma_0 \Rightarrow$ and $\neg\varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

$$
\begin{prooftree}
\AxiomC{$\vdots\ \pi_0$}
\noLine
\UnaryInfC{$\varphi, \Gamma_0 \Rightarrow$}
\RightLabel{$\neg$R}
\UnaryInfC{$\Gamma_0 \Rightarrow \neg\varphi$}
\AxiomC{$\vdots\ \pi_1$}
\noLine
\UnaryInfC{$\neg\varphi, \Gamma_1 \Rightarrow$}
\RightLabel{Cut}
\BinaryInfC{$\Gamma_0, \Gamma_1 \Rightarrow$}
\end{prooftree}
$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence Γ is inconsistent. □

seq.10 Derivability and the Propositional Connectives

explanation We establish that the derivability relation $\vdash$ of the sequent calculus is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \rightarrow \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. fol:seq:ppr:sec

Proposition seq.22. *fol:seq:ppr:prop:provability-land*

1. *Both $\varphi \land \psi \vdash \varphi$ and $\varphi \land \psi \vdash \psi$.* *fol:seq:ppr:prop:provability-land-left*
2. *$\varphi, \psi \vdash \varphi \land \psi$.* *fol:seq:ppr:prop:provability-land-right*

Proof. 1. Both sequents $\varphi \land \psi \Rightarrow \varphi$ and $\varphi \land \psi \Rightarrow \psi$ are derivable:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \land \psi \Rightarrow \varphi}\ \land\mathrm{L} \qquad \dfrac{\psi \Rightarrow \psi}{\varphi \land \psi \Rightarrow \psi}\ \land\mathrm{L}$$

2. Here is a derivation of the sequent $\varphi, \psi \Rightarrow \varphi \land \psi$:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \land \psi}\ \land\mathrm{R}$$

□

Proposition seq.23. *fol:seq:ppr:prop:provability-lor*

1. *$\varphi \lor \psi, \neg\varphi, \neg\psi$ is inconsistent.*
2. *Both $\varphi \vdash \varphi \lor \psi$ and $\psi \vdash \varphi \lor \psi$.*

Proof. 1. We give a derivation of the sequent $\varphi \lor \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\ \neg\mathrm{L}}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow}\ \neg\mathrm{L}}{\psi, \neg\varphi, \neg\psi \Rightarrow}}{\varphi \lor \psi, \neg\varphi, \neg\psi \Rightarrow}\ \lor\mathrm{L}$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \lor \psi$ and $\psi \Rightarrow \varphi \lor \psi$ have derivations:

$$\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \lor \psi}\ \lor\mathrm{R} \qquad \dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \lor \psi}\ \lor\mathrm{R}$$

□

Proposition seq.24. *fol:seq:ppr:prop:provability-lif*

1. *$\varphi, \varphi \rightarrow \psi \vdash \psi$.* *fol:seq:ppr:prop:provability-lif-left*

fol:seq:ppr: prop:provability-lif-right

2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ *and* $\psi \vdash \varphi \rightarrow \psi$.

Proof. 1. The sequent $\varphi \rightarrow \psi, \varphi \Rightarrow \psi$ is derivable:

$$\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi}\ \rightarrow\mathrm{L}$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \rightarrow \psi$ and $\psi \Rightarrow \varphi \rightarrow \psi$ are derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow}\ \neg\mathrm{L}}{\varphi, \neg\varphi \Rightarrow}\ \mathrm{XL}}{\varphi, \neg\varphi \Rightarrow \psi}\ \mathrm{WR}}{\neg\varphi \Rightarrow \varphi \rightarrow \psi}\ \rightarrow\mathrm{R} \qquad \dfrac{\dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}\ \mathrm{WL}}{\psi \Rightarrow \varphi \rightarrow \psi}\ \rightarrow\mathrm{R}$$

□

seq.11 Derivability and the Quantifiers

fol:seq:qpr: sec

The completeness theorem also requires that the sequent calculus rules rules yield the facts about $\vdash$ established in this section. explanation

fol:seq:qpr: thm:strong-generalization

Theorem seq.25. *If c is a constant not occurring in Γ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x\, \varphi(x)$.*

Proof. Let π_0 be an **LK**-derivation of $\Gamma_0 \Rightarrow \varphi(c)$ for some finite $\Gamma_0 \subseteq \Gamma$. By adding a $\forall$R inference, we obtain a derivation of $\Gamma_0 \Rightarrow \forall x\, \varphi(x)$, since c does not occur in Γ or $\varphi(x)$ and thus the eigenvariable condition is satisfied. □

fol:seq:qpr: prop:provability-quantifiers

Proposition seq.26.

1. $\varphi(t) \vdash \exists x\, \varphi(x)$.

2. $\forall x\, \varphi(x) \vdash \varphi(t)$.

Proof. 1. The sequent $\varphi(t) \Rightarrow \exists x\, \varphi(x)$ is derivable:

$$\dfrac{\varphi(t) \Rightarrow \varphi(t)}{\varphi(t) \Rightarrow \exists x\, \varphi(x)}\ \exists\mathrm{R}$$

2. The sequent $\forall x\, \varphi(x) \Rightarrow \varphi(t)$ is derivable:

$$\dfrac{\varphi(t) \Rightarrow \varphi(t)}{\forall x\, \varphi(x) \Rightarrow \varphi(t)}\ \forall\mathrm{L}$$

□

seq.12 Soundness

explanation A derivation system, such as the sequent calculus, is *sound* if it cannot derive things that do not actually hold. Soundness is thus a kind of guaranteed safety property for derivation systems. Depending on which proof theoretic property is in question, we would like to know for instance, that fol:seq:sou:sec

1. every derivable φ is valid;
2. if a sentence is derivable from some others, it is also a consequence of them;
3. if a set of sentences is inconsistent, it is unsatisfiable.

These are important properties of a derivation system. If any of them do not hold, the derivation system is deficient—it would derive too much. Consequently, establishing the soundness of a derivation system is of the utmost importance.

Because all these proof-theoretic properties are defined via derivability in the sequent calculus of certain sequents, proving (1)–(3) above requires proving something about the semantic properties of derivable sequents. We will first define what it means for a sequent to be *valid*, and then show that every derivable sequent is valid. (1)–(3) then follow as corollaries from this result.

Definition seq.27. A structure $\mathfrak{M}$ *satisfies* a sequent $\Gamma \Rightarrow \Delta$ iff either $\mathfrak{M} \nvDash \varphi$ for some $\varphi \in \Gamma$ or $\mathfrak{M} \vDash \varphi$ for some $\varphi \in \Delta$.

A sequent is *valid* iff every structure $\mathfrak{M}$ satisfies it.

Theorem seq.28 (Soundness). *If* **LK** *derives $\Theta \Rightarrow \Xi$, then $\Theta \Rightarrow \Xi$ is valid.* fol:seq:sou:thm:sequent-soundness

Proof. Let π be a derivation of $\Theta \Rightarrow \Xi$. We proceed by induction on the number of inferences n in π.

If the number of inferences is 0, then π consists only of an initial sequent. Every initial sequent $\varphi \Rightarrow \varphi$ is obviously valid, since for every $\mathfrak{M}$, either $\mathfrak{M} \nvDash \varphi$ or $\mathfrak{M} \vDash \varphi$.

If the number of inferences is greater than 0, we distinguish cases according to the type of the lowermost inference. By induction hypothesis, we can assume that the premises of that inference are valid, since the number of inferences in the derivation of any premise is smaller than n.

First, we consider the possible inferences with only one premise.

1. The last inference is a weakening. Then $\Theta \Rightarrow \Xi$ is either $\varphi, \Gamma \Rightarrow \Delta$ (if the last inference is WL) or $\Gamma \Rightarrow \Delta, \varphi$ (if it's WR), and the derivation ends in one of

$$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\text{WL} \qquad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\text{WR}$$

By induction hypothesis, $\Gamma \Rightarrow \Delta$ is valid, i.e., for every structure $\mathfrak{M}$, either there is some $\chi \in \Gamma$ such that $\mathfrak{M} \nvDash \chi$ or there is some $\chi \in \Delta$ such that $\mathfrak{M} \vDash \chi$.

If $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, then $\chi \in \Theta$ as well since $\Theta = \varphi, \Gamma$, and so $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Theta$. Similarly, if $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$, as $\chi \in \Xi$, $\mathfrak{M} \vDash \chi$ for some $\chi \in \Xi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

2. The last inference is $\neg$L: Then the premise of the last inference is $\Gamma \Rightarrow \Delta, \varphi$ and the conclusion is $\neg\varphi, \Gamma \Rightarrow \Delta$, i.e., the derivation ends in

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}\neg\text{L}$$

and $\Theta = \neg\varphi, \Gamma$ while $\Xi = \Delta$.

The induction hypothesis tells us that $\Gamma \Rightarrow \Delta, \varphi$ is valid, i.e., for every $\mathfrak{M}$, either (a) for some $\chi \in \Gamma$, $\mathfrak{M} \nvDash \chi$, or (b) for some $\chi \in \Delta$, $\mathfrak{M} \vDash \chi$, or (c) $\mathfrak{M} \vDash \varphi$. We want to show that $\Theta \Rightarrow \Xi$ is also valid. Let $\mathfrak{M}$ be a structure. If (a) holds, then there is $\chi \in \Gamma$ so that $\mathfrak{M} \nvDash \chi$, but $\chi \in \Theta$ as well. If (b) holds, there is $\chi \in \Delta$ such that $\mathfrak{M} \vDash \chi$, but $\chi \in \Xi$ as well. Finally, if $\mathfrak{M} \vDash \varphi$, then $\mathfrak{M} \nvDash \neg\varphi$. Since $\neg\varphi \in \Theta$, there is $\chi \in \Theta$ such that $\mathfrak{M} \nvDash \chi$. Consequently, $\Theta \Rightarrow \Xi$ is valid.

3. The last inference is $\neg$R: Exercise.

4. The last inference is $\wedge$L: There are two variants: $\varphi \wedge \psi$ may be inferred on the left from φ or from ψ on the left side of the premise. In the first case, the π ends in

$$\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}\wedge\text{L}$$

and $\Theta = \varphi \wedge \psi, \Gamma$ while $\Xi = \Delta$. Consider a structure $\mathfrak{M}$. Since by induction hypothesis, $\varphi, \Gamma \Rightarrow \Delta$ is valid, (a) $\mathfrak{M} \nvDash \varphi$, (b) $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$. In case (a), $\mathfrak{M} \nvDash \varphi \wedge \psi$, so there is $\chi \in \Theta$ (namely, $\varphi \wedge \psi$) such that $\mathfrak{M} \nvDash \chi$. In case (b), there is $\chi \in \Gamma$

such that $\mathfrak{M} \nvDash \chi$, and $\chi \in \Theta$ as well. In case (c), there is $\chi \in \Delta$ such that $\mathfrak{M} \vDash \chi$, and $\chi \in \Xi$ as well since $\Xi = \Delta$. So in each case, $\mathfrak{M}$ satisfies $\varphi \wedge \psi, \Gamma \Rightarrow \Delta$. Since $\mathfrak{M}$ was arbitrary, $\Gamma \Rightarrow \Delta$ is valid. The case where $\varphi \wedge \psi$ is inferred from ψ is handled the same, changing φ to ψ.

5. The last inference is $\vee$R: There are two variants: $\varphi \vee \psi$ may be inferred on the right from φ or from ψ on the right side of the premise. In the first case, π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array}}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee\mathrm{R}$$

Now $\Theta = \Gamma$ and $\Xi = \Delta, \varphi \vee \psi$. Consider a structure $\mathfrak{M}$. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid, (a) $\mathfrak{M} \vDash \varphi$, (b) $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$. In case (a), $\mathfrak{M} \vDash \varphi \vee \psi$. In case (b), there is $\chi \in \Gamma$ such that $\mathfrak{M} \nvDash \chi$. In case (c), there is $\chi \in \Delta$ such that $\mathfrak{M} \vDash \chi$. So in each case, $\mathfrak{M}$ satisfies $\Gamma \Rightarrow \Delta, \varphi \vee \psi$, i.e., $\Theta \Rightarrow \Xi$. Since $\mathfrak{M}$ was arbitrary, $\Theta \Rightarrow \Xi$ is valid. The case where $\varphi \vee \psi$ is inferred from ψ is handled the same, changing φ to ψ.

6. The last inference is $\rightarrow$R: Then π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \varphi, \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ \rightarrow\mathrm{R}$$

Again, the induction hypothesis says that the premise is valid; we want to show that the conclusion is valid as well. Let $\mathfrak{M}$ be arbitrary. Since $\varphi, \Gamma \Rightarrow \Delta, \psi$ is valid, at least one of the following cases obtains: (a) $\mathfrak{M} \nvDash \varphi$, (b) $\mathfrak{M} \vDash \psi$, (c) $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, or (d) $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$. In cases (a) and (b), $\mathfrak{M} \vDash \varphi \rightarrow \psi$ and so there is a $\chi \in \Delta, \varphi \rightarrow \psi$ such that $\mathfrak{M} \vDash \chi$. In case (c), for some $\chi \in \Gamma$, $\mathfrak{M} \nvDash \chi$. In case (d), for some $\chi \in \Delta$, $\mathfrak{M} \vDash \chi$. In each case, $\mathfrak{M}$ satisfies $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$. Since $\mathfrak{M}$ was arbitrary, $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$ is valid.

7. The last inference is $\forall$L: Then there is a formula $\varphi(x)$ and a closed term t such that π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \varphi(t), \Gamma \Rightarrow \Delta\end{array}}{\forall x\, \varphi(x), \Gamma \Rightarrow \Delta}\ \forall\mathrm{L}$$

We want to show that the conclusion $\forall x\, \varphi(x), \Gamma \Rightarrow \Delta$ is valid. Consider a structure $\mathfrak{M}$. Since the premise $\varphi(t), \Gamma \Rightarrow \Delta$ is valid, (a) $\mathfrak{M} \nvDash \varphi(t)$, (b) $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$. In case (a), by **??**, if $\mathfrak{M} \vDash \forall x\, \varphi(x)$, then $\mathfrak{M} \vDash \varphi(t)$. Since $\mathfrak{M} \nvDash \varphi(t)$, $\mathfrak{M} \nvDash \forall x\, \varphi(x)$. In case (b) and (c), $\mathfrak{M}$ also satisfies $\forall x\, \varphi(x), \Gamma \Rightarrow \Delta$. Since $\mathfrak{M}$ was arbitrary, $\forall x\, \varphi(x), \Gamma \Rightarrow \Delta$ is valid.

8. The last inference is $\exists$R: Exercise.

9. The last inference is $\forall$R: Then there is a formula $\varphi(x)$ and a constant symbol a such that π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi(a)\end{array}}{\Gamma \Rightarrow \Delta, \forall x\, \varphi(x)}\ \forall\text{R}$$

where the eigenvariable condition is satisfied, i.e., a does not occur in $\varphi(x)$, Γ, or Δ. By induction hypothesis, the premise of the last inference is valid. We have to show that the conclusion is valid as well, i.e., that for any structure $\mathfrak{M}$, (a) $\mathfrak{M} \vDash \forall x\, \varphi(x)$, (b) $\mathfrak{M} \nvDash \chi$ for some $\chi \in \Gamma$, or (c) $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$.

Suppose $\mathfrak{M}$ is an arbitrary structure. If (b) or (c) holds, we are done, so suppose neither holds: for all $\chi \in \Gamma$, $\mathfrak{M} \vDash \chi$, and for all $\chi \in \Delta$, $\mathfrak{M} \nvDash \chi$. We have to show that (a) holds, i.e., $\mathfrak{M} \vDash \forall x\, \varphi(x)$. By **??**, if suffices to show that $\mathfrak{M}, s \vDash \varphi(x)$ for all variable assignments s. So let s be an arbitrary variable assignment. Consider the structure $\mathfrak{M}'$ which is just like $\mathfrak{M}$ except $a^{\mathfrak{M}'} = s(x)$. By **??**, for any $\chi \in \Gamma$, $\mathfrak{M}' \vDash \chi$ since a does not occur in Γ, and for any $\chi \in \Delta$, $\mathfrak{M}' \nvDash \chi$. But the premise is valid, so $\mathfrak{M}' \vDash \varphi(a)$. By **??**, $\mathfrak{M}', s \vDash \varphi(a)$, since $\varphi(a)$ is a sentence. Now $s \sim_x s$ with $s(x) = \mathrm{Val}_s^{\mathfrak{M}'}(a)$, since we've defined $\mathfrak{M}'$ in just this way. So **??** applies, and we get $\mathfrak{M}', s \vDash \varphi(x)$. Since a does not occur in $\varphi(x)$, by **??**, $\mathfrak{M}, s \vDash \varphi(x)$. Since s was arbitrary, we've completed the proof that $\mathfrak{M}, s \vDash \varphi(x)$ for all variable assignments.

10. The last inference is $\exists$L: Exercise.

Now let's consider the possible inferences with two premises.

1. The last inference is a cut: then π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \varphi, \Pi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \text{Cut}$$

Let $\mathfrak{M}$ be a structure. By induction hypothesis, the premises are valid, so $\mathfrak{M}$ satisfies both premises. We distinguish two cases: (a) $\mathfrak{M} \nvDash \varphi$ and (b) $\mathfrak{M} \vDash \varphi$. In case (a), in order for $\mathfrak{M}$ to satisfy the left premise, it must satisfy $\Gamma \Rightarrow \Delta$. But then it also satisfies the conclusion. In case (b), in order for $\mathfrak{M}$ to satisfy the right premise, it must satisfy $\Pi \setminus \Lambda$. Again, $\mathfrak{M}$ satisfies the conclusion.

2. The last inference is $\wedge$R. Then π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \psi\end{array}}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge\mathrm{R}$$

   Consider a structure $\mathfrak{M}$. If $\mathfrak{M}$ satisfies $\Gamma \Rightarrow \Delta$, we are done. So suppose it doesn't. Since $\Gamma \Rightarrow \Delta, \varphi$ is valid by induction hypothesis, $\mathfrak{M} \vDash \varphi$. Similarly, since $\Gamma \Rightarrow \Delta, \psi$ is valid, $\mathfrak{M} \vDash \psi$. But then $\mathfrak{M} \vDash \varphi \wedge \psi$.

3. The last inference is $\vee$L: Exercise.

4. The last inference is $\rightarrow$L. Then π ends in

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, \varphi\end{array} \qquad \begin{array}{c}\vdots\\ \psi, \Pi \Rightarrow \Lambda\end{array}}{\varphi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda} \rightarrow\mathrm{L}$$

   Again, consider a structure $\mathfrak{M}$ and suppose $\mathfrak{M}$ doesn't satisfy $\Gamma, \Pi \Rightarrow \Delta, \Lambda$. We have to show that $\mathfrak{M} \nvDash \varphi \rightarrow \psi$. If $\mathfrak{M}$ doesn't satisfy $\Gamma, \Pi \Rightarrow \Delta, \Lambda$, it satisfies neither $\Gamma \Rightarrow \Delta$ nor $\Pi \Rightarrow \Lambda$. Since, $\Gamma \Rightarrow \Delta, \varphi$ is valid, we have $\mathfrak{M} \vDash \varphi$. Since $\psi, \Pi \Rightarrow \Lambda$ is valid, we have $\mathfrak{M} \nvDash \psi$. But then $\mathfrak{M} \nvDash \varphi \rightarrow \psi$, which is what we wanted to show. □

Problem seq.8. Complete the proof of Theorem seq.28.

Corollary seq.29. *If $\vdash \varphi$ then φ is valid.* fol:seq:sou: cor:weak-soundness

Corollary seq.30. *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.* fol:seq:sou: cor:entailment-soundness

Proof. If $\Gamma \vdash \varphi$ then for some finite subset $\Gamma_0 \subseteq \Gamma$, there is a derivation of $\Gamma_0 \Rightarrow \varphi$. By Theorem seq.28, every structure $\mathfrak{M}$ either makes some $\psi \in \Gamma_0$ false or makes φ true. Hence, if $\mathfrak{M} \vDash \Gamma$ then also $\mathfrak{M} \vDash \varphi$. □

Corollary seq.31. *If Γ is satisfiable, then it is consistent.* fol:seq:sou: cor:consistency-soundness

Proof. We prove the contrapositive. Suppose that Γ is not consistent. Then there is a finite $\Gamma_0 \subseteq \Gamma$ and a derivation of $\Gamma_0 \Rightarrow \quad$. By Theorem seq.28, $\Gamma_0 \Rightarrow \quad$ is valid. In other words, for every structure $\mathfrak{M}$, there is $\chi \in \Gamma_0$ so that $\mathfrak{M} \nvDash \chi$, and since $\Gamma_0 \subseteq \Gamma$, that χ is also in Γ. Thus, no $\mathfrak{M}$ satisfies Γ, and Γ is not satisfiable. □

seq.13 Derivations with Identity predicate

fol:seq:ide:sec Derivations with identity predicate require additional initial sequents and inference rules.

Definition seq.32 (Initial sequents for =). If t is a closed term, then $\Rightarrow t = t$ is an initial sequent.

The rules for $=$ are (t_1 and t_2 are closed terms):

$$\dfrac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)} = \qquad\qquad \dfrac{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)}{t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)} =$$

Example seq.33. If s and t are closed terms, then $s = t, \varphi(s) \vdash \varphi(t)$:

$$\dfrac{\dfrac{\varphi(s) \Rightarrow \varphi(s)}{s = t, \varphi(s) \Rightarrow \varphi(s)}\,\text{WL}}{s = t, \varphi(s) \Rightarrow \varphi(t)} =$$

This may be familiar as the principle of substitutability of identicals, or Leibniz' Law.

LK proves that $=$ is symmetric and transitive:

$$\dfrac{\dfrac{\Rightarrow t_1 = t_1}{t_1 = t_2 \Rightarrow t_1 = t_1}\,\text{WL}}{t_1 = t_2 \Rightarrow t_2 = t_1} = \qquad\qquad \dfrac{\dfrac{\dfrac{t_1 = t_2 \Rightarrow t_1 = t_2}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_2}\,\text{WL}}{t_2 = t_3, t_1 = t_2 \Rightarrow t_1 = t_3} =}{t_1 = t_2, t_2 = t_3 \Rightarrow t_1 = t_3}\,\text{XL}$$

In the derivation on the left, the formula $x = t_1$ is our $\varphi(x)$. On the right, we take $\varphi(x)$ to be $t_1 = x$.

Problem seq.9. Give derivations of the following sequents:

1. $\Rightarrow \forall x\, \forall y\, ((x = y \land \varphi(x)) \rightarrow \varphi(y))$

2. $\exists x\, \varphi(x) \land \forall y\, \forall z\, ((\varphi(y) \land \varphi(z)) \rightarrow y = z) \Rightarrow \exists x\, (\varphi(x) \land \forall y\, (\varphi(y) \rightarrow y = x))$

seq.14 Soundness with Identity predicate

Proposition seq.34. **LK** *with initial sequents and rules for identity is sound.* fol:seq:sid:sec

Proof. Initial sequents of the form $\Rightarrow t = t$ are valid, since for every structure $\mathfrak{M}$, $\mathfrak{M} \vDash t = t$. (Note that we assume the term t to be closed, i.e., it contains no variables, so variable assignments are irrelevant).

Suppose the last inference in a derivation is $=$. Then the premise is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_1)$ and the conclusion is $t_1 = t_2, \Gamma \Rightarrow \Delta, \varphi(t_2)$. Consider a structure $\mathfrak{M}$. We need to show that the conclusion is valid, i.e., if $\mathfrak{M} \vDash t_1 = t_2$ and $\mathfrak{M} \vDash \Gamma$, then either $\mathfrak{M} \vDash \chi$ for some $\chi \in \Delta$ or $\mathfrak{M} \vDash \varphi(t_2)$.

By induction hypothesis, the premise is valid. This means that if $\mathfrak{M} \vDash t_1 = t_2$ and $\mathfrak{M} \vDash \Gamma$ either (a) for some $\chi \in \Delta$, $\mathfrak{M} \vDash \chi$ or (b) $\mathfrak{M} \vDash \varphi(t_1)$. In case (a) we are done. Consider case (b). Let s be a variable assignment with $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_1)$. By **??**, $\mathfrak{M}, s \vDash \varphi(t_1)$. Since $s \sim_x s$, by **??**, $\mathfrak{M}, s \vDash \varphi(x)$. since $\mathfrak{M} \vDash t_1 = t_2$, we have $\mathrm{Val}^{\mathfrak{M}}(t_1) = \mathrm{Val}^{\mathfrak{M}}(t_2)$, and hence $s(x) = \mathrm{Val}^{\mathfrak{M}}(t_2)$. By applying **??** again, we also have $\mathfrak{M}, s \vDash \varphi(t_2)$. By **??**, $\mathfrak{M} \vDash \varphi(t_2)$. □

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