seq.1  Quantifier Rules

Rules for \( \forall \)

\[
\begin{align*}
\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} & \quad \forall L \\
\frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} & \quad \forall R
\end{align*}
\]

In \( \forall L \), \( t \) is a closed term (i.e., one without variables). In \( \forall R \), \( a \) is a constant symbol which must not occur anywhere in the lower sequent of the \( \forall R \) rule. We call \( a \) the eigenvariable of the \( \forall R \) inference.

Rules for \( \exists \)

\[
\begin{align*}
\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} & \quad \exists L \\
\frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} & \quad \exists R
\end{align*}
\]

Again, \( t \) is a closed term, and \( a \) is a constant symbol which does not occur in the lower sequent of the \( \exists L \) rule. We call \( a \) the eigenvariable of the \( \exists L \) inference.

The condition that an eigenvariable not occur in the lower sequent of the \( \forall R \) or \( \exists L \) inference is called the eigenvariable condition.

Recall the convention that when \( \varphi \) is a formula with the variable \( x \) free, we indicate this by writing \( \varphi(x) \). In the same context, \( \varphi(t) \) then is short for \( \varphi[t/x] \).

So we could also write the \( \exists R \) rule as:

\[
\frac{\Gamma \Rightarrow \Delta, \varphi[t/x]}{\Gamma \Rightarrow \Delta, \exists x \varphi} \quad \exists R
\]

Note that \( t \) may already occur in \( \varphi \), e.g., \( \varphi \) might be \( P(t, x) \). Thus, inferring \( \Gamma \Rightarrow \Delta, \exists x P(t, x) \) from \( \Gamma \Rightarrow \Delta, P(t, t) \) is a correct application of \( \exists R \)—you may “replace” one or more, and not necessarily all, occurrences of \( t \) in the premise by the bound variable \( x \). However, the eigenvariable conditions in \( \forall R \) and \( \exists L \) require that the constant symbol \( a \) does not occur in \( \varphi \). So, you cannot correctly infer \( \Gamma \Rightarrow \Delta, \forall x P(a, x) \) from \( \Gamma \Rightarrow \Delta, P(a, a) \) using \( \forall R \).

In \( \exists R \) and \( \forall L \) there are no restrictions on the term \( t \). On the other hand, in the \( \exists L \) and \( \forall R \) rules, the eigenvariable condition requires that the constant symbol \( a \) does not occur anywhere outside of \( \varphi(a) \) in the upper sequent. It is necessary to ensure that the system is sound, i.e., only derives sequents that are valid. Without this condition, the following would be allowed:

---

\(^1\)We use the term “eigenvariable” even though \( a \) in the above rule is a constant symbol. This has historical reasons.
\[
\varphi(a) \Rightarrow \varphi(a) \\
\exists x \varphi(x) \Rightarrow \varphi(a) \quad \exists L \\
\exists x \varphi(x) \Rightarrow \forall x \varphi(x) \quad \forall R \\
\varphi(a) \Rightarrow \varphi(a) \\
\exists x \varphi(x) \Rightarrow \forall x \varphi(x) \quad \exists L \\
\exists x \varphi(x) \Rightarrow \forall x \varphi(x) \quad \forall R \\
\exists x \varphi(x) \Rightarrow \forall x \varphi(x) \quad \exists L
\]

However, \(\exists x \varphi(x) \Rightarrow \forall x \varphi(x)\) is not valid.

Photo Credits

Bibliography