seq.1  Derivability and the Quantifiers

The completeness theorem also requires that the sequent calculus rules rules yield the facts about $\vdash$ established in this section.

**Theorem seq.1.** If $c$ is a constant not occurring in $\Gamma$ or $\varphi(x)$ and $\Gamma \vdash \varphi(c)$, then $\Gamma \vdash \forall x \varphi(x)$.

**Proof.** Let $\pi_0$ be an LK-derivation of $\Gamma_0 \Rightarrow \varphi(c)$ for some finite $\Gamma_0 \subseteq \Gamma$. By adding a $\forall R$ inference, we obtain a derivation of $\Gamma_0 \Rightarrow \forall x \varphi(x)$, since $c$ does not occur in $\Gamma$ or $\varphi(x)$ and thus the eigenvariable condition is satisfied. \(\square\)

**Proposition seq.2.**

1. $\varphi(t) \vdash \exists x \varphi(x)$.
2. $\forall x \varphi(x) \vdash \varphi(t)$.

**Proof.**

1. The sequent $\varphi(t) \Rightarrow \exists x \varphi(x)$ is derivable:

$$
\begin{array}{c}
\varphi(t) \Rightarrow \varphi(t) \\
\frac{}{\varphi(t) \Rightarrow \exists x \varphi(x)} \exists R
\end{array}
$$

2. The sequent $\forall x \varphi(x) \Rightarrow \varphi(t)$ is derivable:

$$
\begin{array}{c}
\varphi(t) \Rightarrow \varphi(t) \\
\frac{}{\forall x \varphi(x) \Rightarrow \varphi(t)} \forall L
\end{array}
$$

\(\square\)

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