

seq.1 Derivability and the Propositional Connectives

fol:seq:ppr: sec We establish that the **derivability** relation \vdash of the sequent calculus is strong explanation enough to establish some basic facts involving the propositional connectives, such as that $\varphi \wedge \psi \vdash \varphi$ and $\varphi, \varphi \rightarrow \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem.

Proposition seq.1.

fol:seq:ppr: prop:provability-land
fol:seq:ppr: prop:provability-land-left
fol:seq:ppr: prop:provability-land-right

1. Both $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$.
2. $\varphi, \psi \vdash \varphi \wedge \psi$.

Proof. 1. Both sequents $\varphi \wedge \psi \Rightarrow \varphi$ and $\varphi \wedge \psi \Rightarrow \psi$ are **derivable**:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi} \wedge L \quad \frac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi} \wedge L$$

2. Here is a **derivation** of the sequent $\varphi, \psi \Rightarrow \varphi \wedge \psi$:

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi \wedge \psi} \wedge R \quad \square$$

Proposition seq.2.

fol:seq:ppr: prop:provability-lor

1. $\varphi \vee \psi, \neg\varphi, \neg\psi$ is inconsistent.
2. Both $\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$.

Proof. 1. We give a **derivation** of the sequent $\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow$:

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg L}{\varphi, \neg\varphi, \neg\psi \Rightarrow} \quad \frac{\frac{\psi \Rightarrow \psi}{\neg\psi, \psi \Rightarrow} \neg L}{\psi, \neg\varphi, \neg\psi \Rightarrow} \neg L}{\varphi \vee \psi, \neg\varphi, \neg\psi \Rightarrow} \vee L$$

(Recall that double inference lines indicate several weakening, contraction, and exchange inferences.)

2. Both sequents $\varphi \Rightarrow \varphi \vee \psi$ and $\psi \Rightarrow \varphi \vee \psi$ have **derivations**:

$$\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi} \vee R \quad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi} \vee R \quad \square$$

Proposition seq.3.

fol:seq:ppr: prop:provability-lif
fol:seq:ppr: prop:provability-lif-left

1. $\varphi, \varphi \rightarrow \psi \vdash \psi$.

2. Both $\neg\varphi \vdash \varphi \rightarrow \psi$ and $\psi \vdash \varphi \rightarrow \psi$.

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prop:provability-lif-right*

Proof. 1. The sequent $\varphi \rightarrow \psi, \varphi \Rightarrow \psi$ is **derivable**:

$$\frac{\varphi \Rightarrow \varphi \quad \psi \Rightarrow \psi}{\varphi \rightarrow \psi, \varphi \Rightarrow \psi} \rightarrow L$$

2. Both sequents $\neg\varphi \Rightarrow \varphi \rightarrow \psi$ and $\psi \Rightarrow \varphi \rightarrow \psi$ are **derivable**:

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\neg\varphi, \varphi \Rightarrow} \neg L}{\varphi, \neg\varphi \Rightarrow} XL}{\varphi, \neg\varphi \Rightarrow \psi} WR \quad \frac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi} WL \rightarrow R$$

$$\frac{\varphi, \neg\varphi \Rightarrow \psi}{\neg\varphi \Rightarrow \varphi \rightarrow \psi} \rightarrow R \quad \frac{\psi \Rightarrow \varphi \rightarrow \psi}{\psi \Rightarrow \varphi \rightarrow \psi} \rightarrow R \quad \square$$

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Bibliography