We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

**Proposition seq.1.** If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

*Proof.* There are finite $\Gamma_0$ and $\Gamma_1 \subseteq \Gamma$ such that LK derives $\Gamma_0 \Rightarrow \varphi$ and $\varphi, \Gamma_1 \Rightarrow \varphi$. Let the LK-derivation of $\Gamma_0 \Rightarrow \varphi$ be $\pi_0$ and the LK-derivation of $\varphi, \Gamma_1 \Rightarrow \varphi$ be $\pi_1$. We can then derive

$$
\begin{array}{c}
\vdots \pi_0 \\
\vdots \pi_1 \\
\end{array}
\Gamma_0 \Rightarrow \varphi \quad \varphi, \Gamma_1 \Rightarrow \varphi
\overset{\text{Cut}}{=} \frac{\Gamma_0, \Gamma_1 \Rightarrow \varphi}{\text{Cut}}
\end{array}
$$

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$, hence $\Gamma$ is inconsistent. $\Box$

**Proposition seq.2.** $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

*Proof.* First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\pi_0$ of $\Gamma \Rightarrow \varphi$. By adding a $\neg$ rule, we obtain a derivation of $\neg \varphi, \Gamma \Rightarrow \varphi$, i.e., $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

If $\Gamma \cup \{\neg \varphi\}$ is inconsistent, there is a derivation $\pi_1$ of $\neg \varphi, \Gamma \Rightarrow \varphi$. The following is a derivation of $\Gamma \Rightarrow \varphi$:

$$
\begin{array}{c}
\vdots \pi_1 \\
\varphi \Rightarrow \varphi \\
\Rightarrow \varphi, \neg \varphi \\
\neg \varphi, \Gamma \Rightarrow \varphi
\overset{\text{Cut}}{=} \frac{\Gamma \Rightarrow \varphi}{\text{Cut}}
\end{array}
$$

$\Box$

**Problem seq.1.** Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

**Proposition seq.3.** If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

*Proof.* Suppose $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$. Then there is a derivation $\pi$ of a sequent $\Gamma_0 \Rightarrow \varphi$. The sequent $\neg \varphi, \Gamma_0 \Rightarrow \varphi$ is also derivable:

$$
\begin{array}{c}
\vdots \pi \\
\vdots \neg \varphi, \varphi \Rightarrow \varphi \\
\neg \varphi \Rightarrow \varphi \\
\Gamma_0 \Rightarrow \varphi
\overset{\text{Cut}}{=} \frac{\Gamma_0, \neg \varphi \Rightarrow \varphi}{\text{Cut}}
\end{array}
$$

Since $\neg \varphi \in \Gamma$ and $\Gamma_0 \subseteq \Gamma$, this shows that $\Gamma$ is inconsistent. $\Box$
Proposition seq.4. If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

Proof. There are finite sets $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$ and LK-derivations $\pi_0$ and $\pi_1$ of $\varphi, \Gamma_0 \Rightarrow$ and $\neg \varphi, \Gamma_1 \Rightarrow$, respectively. We can then derive

\[
\begin{array}{c}
\vdots \\
\varphi, \Gamma_0 \Rightarrow \\
\vdots \\
I_0 \Rightarrow \neg \varphi \\
\vdots \\
\neg \varphi, \Gamma_1 \Rightarrow \\
\vdots \\
\Gamma_0, \Gamma_1 \Rightarrow \\
\end{array}
\]

Cut

Since $\Gamma_0 \subseteq \Gamma$ and $\Gamma_1 \subseteq \Gamma$, $\Gamma_0 \cup \Gamma_1 \subseteq \Gamma$. Hence $\Gamma$ is inconsistent. $\square$

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Bibliography