

This section collects the definitions of the provability relation and consistency for natural deduction.

seq.1 Proof-Theoretic Notions

fol:seq:ptn:
sec

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain sequents. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorem*.

explanation

Definition seq.1 (Theorems). A sentence φ is a *theorem* if there is a **derivation** in **LK** of the sequent $\Rightarrow \varphi$. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition seq.2 (Derivability). A sentence φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ and a sequence Γ'_0 of the **sentences** in Γ_0 such that **LK** **derives** $\Gamma'_0 \Rightarrow \varphi$. If φ is not **derivable** from Γ we write $\Gamma \nvdash \varphi$.

Because of the contraction, weakening, and exchange rules, the order and number of **sentences** in Γ'_0 does not matter: if a sequent $\Gamma'_0 \Rightarrow \varphi$ is **derivable**, then so is $\Gamma''_0 \Rightarrow \varphi$ for any Γ''_0 that contains the same **sentences** as Γ'_0 . For instance, if $\Gamma_0 = \{\psi, \chi\}$ then both $\Gamma'_0 = \langle \psi, \psi, \chi \rangle$ and $\Gamma''_0 = \langle \chi, \chi, \psi \rangle$ are sequences containing just the **sentences** in Γ_0 . If a sequent containing one is **derivable**, so is the other, e.g.:

$$\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \frac{\psi, \psi, \chi \Rightarrow \varphi}{\psi, \chi \Rightarrow \varphi} \text{ CL} \\ \frac{\psi, \chi \Rightarrow \varphi}{\chi, \psi \Rightarrow \varphi} \text{ XL} \\ \frac{\chi, \psi \Rightarrow \varphi}{\chi, \chi, \psi \Rightarrow \varphi} \text{ WL} \end{array}$$

From now on we'll say that if Γ_0 is a finite set of **sentences** then $\Gamma_0 \Rightarrow \varphi$ is any sequent where the antecedent is a sequence of **sentences** in Γ_0 and tacitly include contractions, exchanges, and weakenings if necessary.

Definition seq.3 (Consistency). A set of sentences Γ is *inconsistent* iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** **derives** $\Gamma_0 \Rightarrow \cdot$. If Γ is not inconsistent, i.e., if for every finite $\Gamma_0 \subseteq \Gamma$, **LK** does not **derive** $\Gamma_0 \Rightarrow \cdot$, we say it is *consistent*.

fol:seq:ptn:
prop:reflexivity

Proposition seq.4 (Reflexivity). If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Proof. The initial sequent $\varphi \Rightarrow \varphi$ is **derivable**, and $\{\varphi\} \subseteq \Gamma$. □

Proposition seq.5 (Monotonicity). *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

*fol:seq:ptn:
prop:monotonicity*

Proof. Suppose $\Gamma \vdash \varphi$, i.e., there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \Rightarrow \varphi$ is **derivable**. Since $\Gamma \subseteq \Delta$, then Γ_0 is also a finite subset of Δ . The **derivation** of $\Gamma_0 \Rightarrow \varphi$ thus also shows $\Delta \vdash \varphi$. □

Proposition seq.6 (Transitivity). *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

*fol:seq:ptn:
prop:transitivity*

Proof. If $\Gamma \vdash \varphi$, there is a finite $\Gamma_0 \subseteq \Gamma$ and a **derivation** π_0 of $\Gamma_0 \Rightarrow \varphi$. If $\{\varphi\} \cup \Delta \vdash \psi$, then for some finite subset $\Delta_0 \subseteq \Delta$, there is a **derivation** π_1 of $\varphi, \Delta_0 \Rightarrow \psi$. Consider the following **derivation**:

$$\frac{\begin{array}{c} \vdots \\ \vdots \pi_0 \\ \vdots \\ \Gamma_0 \Rightarrow \varphi \end{array} \quad \begin{array}{c} \vdots \\ \vdots \pi_1 \\ \vdots \\ \varphi, \Delta_0 \Rightarrow \psi \end{array}}{\Gamma_0, \Delta_0 \Rightarrow \psi} \text{Cut}$$

Since $\Gamma_0 \cup \Delta_0 \subseteq \Gamma \cup \Delta$, this shows $\Gamma \cup \Delta \vdash \psi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition seq.7. *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every sentence φ .*

*fol:seq:ptn:
prop:incons*

Proof. Exercise. □

Problem seq.1. Prove **Proposition seq.7**

Proposition seq.8 (Compactness).

*fol:seq:ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ has a **derivation**. Consequently, $\Gamma_0 \vdash \varphi$.

2. If Γ is inconsistent, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that **LK** derives $\Gamma_0 \Rightarrow$. But then Γ_0 is a finite subset of Γ that is inconsistent. □

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Bibliography