prf.1 Tableaux

fol:prf:tab:sec While many derivation systems operate with arrangements of sentences, tableaux operate with signed formulas. A signed formula is a pair consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a sentence

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

A tableau consists of signed formulas arranged in a downward-branching tree. It begins with a number of *assumptions* and continues with signed formulas which result from one of the signed formulas above it by applying one of the rules of inference. Each rule allows us to add one or more signed formulas to the end of a branch, or two signed formulas side by side—in this case a branch splits into two, with the two added signed formulas forming the ends of the two branches.

A rule applied to a complex signed formula results in the addition of signed formulas which are immediate sub-formulas. They come in pairs, one rule for each of the two signs. For instance, the $\wedge\mathbb{T}$ rule applies to $\mathbb{T}\,\varphi \wedge \psi$, and allows the addition of both the two signed formulas $\mathbb{T}\,\varphi$ and $\mathbb{T}\,\psi$ to the end of any branch containing $\mathbb{T}\,\varphi \wedge \psi$, and the rule $\varphi \wedge \psi\mathbb{F}$ allows a branch to be split by adding $\mathbb{F}\,\varphi$ and $\mathbb{F}\,\psi$ side-by-side. A tableau is closed if every one of its branches contains a matching pair of signed formulas $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$.

The $\vdash$ relation based on tableaux is defined as follows: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed tableau for the assumptions

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

For instance, here is a closed tableau that shows that $\vdash (\varphi \wedge \psi) \to \varphi$:

1.	$\mathbb{F}\,(\varphi \wedge \psi) \to \varphi$	Assumption
2.	$\mathbb{T}\,\varphi \wedge \psi$	$\to\mathbb{F}\,1$
3.	$\mathbb{F}\,\varphi$	$\to\mathbb{F}\,1$
4.	$\mathbb{T}\,\varphi$	$\to\mathbb{T}\,2$
5.	$\mathbb{T}\,\psi$	$\to\mathbb{T}\,2$
	$\otimes$	

A set Γ is inconsistent in the tableau calculus if there is a closed tableau for assumptions

$$\{\mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}$$

for some $\psi_i \in \Gamma$.

Tableaux were invented in the 1950s independently by Evert Beth and Jaakko Hintikka, and simplified and popularized by Raymond Smullyan. They are very easy to use, since constructing a tableau is a very systematic procedure. Because of the systematic nature of tableaux, they also lend themselves to implementation by computer. However, a tableau is often hard to read and their connection to proofs are sometimes not easy to see. The approach is also quite general, and many different logics have tableau systems. Tableaux also help us to find structures that satisfy given (sets of) sentences: if the set is satisfiable, it won't have a closed tableau, i.e., any tableau will have an open branch. The satisfying structure can be "read off" an open branch, provided every rule it is possible to apply has been applied on that branch. There is also a very close connection to the sequent calculus: essentially, a closed tableau is a condensed derivation in the sequent calculus, written upside-down.

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Bibliography