

prf.1 Natural Deduction

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Natural deduction is a **derivation** system intended to mirror actual reasoning (especially the kind of regimented reasoning employed by mathematicians). Actual reasoning proceeds by a number of “natural” patterns. For instance, proof by cases allows us to establish a conclusion on the basis of a disjunctive premise, by establishing that the conclusion follows from either of the disjuncts. Indirect proof allows us to establish a conclusion by showing that its negation leads to a contradiction. Conditional proof establishes a conditional claim “if ... then ...” by showing that the consequent follows from the antecedent. Natural deduction is a formalization of some of these natural inferences. Each of the logical connectives and quantifiers comes with two rules, an introduction and an elimination rule, and they each correspond to one such natural inference pattern. For instance, \rightarrow Intro corresponds to conditional proof, and \vee Elim to proof by cases. A particularly simple rule is \wedge Elim which allows the inference from $\varphi \wedge \psi$ to φ (or ψ).

One feature that distinguishes natural deduction from other **derivation** systems is its use of assumptions. A **derivation** in natural deduction is a tree of **formulas**. A single **formula** stands at the root of the tree of **formulas**, and the “leaves” of the tree are **formulas** from which the conclusion is derived. In natural deduction, some leaf **formulas** play a role inside the **derivation** but are “used up” by the time the **derivation** reaches the conclusion. This corresponds to the practice, in actual reasoning, of introducing hypotheses which only remain in effect for a short while. For instance, in a proof by cases, we assume the truth of each of the disjuncts; in conditional proof, we assume the truth of the antecedent; in indirect proof, we assume the truth of the negation of the conclusion. This way of introducing hypothetical assumptions and then doing away with them in the service of establishing an intermediate step is a hallmark of natural deduction. The formulas at the leaves of a natural deduction **derivation** are called assumptions, and some of the rules of inference may “**discharge**” them. For instance, if we have a **derivation** of ψ from some assumptions which include φ , then the \rightarrow Intro rule allows us to infer $\varphi \rightarrow \psi$ and discharge any assumption of the form φ . (To keep track of which assumptions are discharged at which inferences, we label the inference and the assumptions it discharges with a number.) The assumptions that remain **undischarged** at the end of the **derivation** are together sufficient for the truth of the conclusion, and so a **derivation** establishes that its **undischarged** assumptions entail its conclusion.

The relation $\Gamma \vdash \varphi$ based on natural deduction holds iff there is a **derivation** in which φ is the last **sentence** in the tree, and every leaf which is **undischarged** is in Γ . φ is a theorem in natural deduction iff there is a **derivation** in which φ is the last **sentence** and all assumptions are **discharged**. For instance, here is a **derivation** that shows that $\vdash (\varphi \wedge \psi) \rightarrow \varphi$:

$$1 \frac{\frac{[\varphi \wedge \psi]^1}{\varphi} \wedge \text{Elim}}{(\varphi \wedge \psi) \rightarrow \varphi} \rightarrow \text{Intro}$$

The label 1 indicates that the assumption $\varphi \wedge \psi$ is **discharged** at the \rightarrow Intro inference.

A set Γ is inconsistent iff $\Gamma \vdash \perp$ in natural deduction. The rule \perp_I makes it so that from an inconsistent set, any **sentence** can be **derived**.

Natural deduction systems were developed by Gerhard Gentzen and Stanisław Jaśkowski in the 1930s, and later developed by Dag Prawitz and Frederic Fitch. Because its inferences mirror natural methods of proof, it is favored by philosophers. The versions developed by Fitch are often used in introductory logic textbooks. In the philosophy of logic, the rules of natural deduction have sometimes been taken to give the meanings of the logical operators (“proof-theoretic semantics”).

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Bibliography