

ntd.1 Quantifier Rules

fol:ntd:qrl: Rules for \forall sec

$$\frac{\varphi(a)}{\forall x \varphi(x)} \forall\text{Intro} \qquad \frac{\forall x \varphi(x)}{\varphi(t)} \forall\text{Elim}$$

In the rules for \forall , t is a closed term (a term that does not contain any variables), and a is a **constant symbol** which does not occur in the conclusion $\forall x \varphi(x)$, or in any assumption which is **undischarged** in the **derivation** ending with the premise $\varphi(a)$. We call a the *eigenvariable* of the $\forall\text{Intro}$ inference.¹

Rules for \exists

$$\frac{\varphi(t)}{\exists x \varphi(x)} \exists\text{Intro} \qquad \begin{array}{c} [\varphi(a)]^n \\ \vdots \\ \vdots \\ \chi \end{array} \exists\text{Elim}$$

Again, t is a closed term, and a is a constant which does not occur in the premise $\exists x \varphi(x)$, in the conclusion χ , or any assumption which is **undischarged** in the **derivations** ending with the two premises (other than the assumptions $\varphi(a)$). We call a the *eigenvariable* of the $\exists\text{Elim}$ inference.

The condition that an eigenvariable neither occur in the premises nor in any assumption that is **undischarged** in the **derivations** leading to the premises for the $\forall\text{Intro}$ or $\exists\text{Elim}$ inference is called the *eigenvariable condition*.

Recall the convention that when φ is a **formula** with the **variable** x free, we indicate this by writing $\varphi(x)$. In the same context, $\varphi(t)$ then is short for $\varphi[t/x]$. So we could also write the $\exists\text{Intro}$ rule as:

$$\frac{\varphi[t/x]}{\exists x \varphi} \exists\text{Intro}$$

Note that t may already occur in φ , e.g., φ might be $P(t, x)$. Thus, inferring $\exists x P(t, x)$ from $P(t, t)$ is a correct application of $\exists\text{Intro}$ —you may “replace” one or more, and not necessarily all, occurrences of t in the premise by the bound **variable** x . However, the eigenvariable conditions in $\forall\text{Intro}$ and $\exists\text{Elim}$ require that the **constant symbol** a does not occur in φ . So, you cannot correctly infer $\forall x P(a, x)$ from $P(a, a)$ using $\forall\text{Intro}$.

¹We use the term “eigenvariable” even though a in the above rule is a constant. This has historical reasons.

explanation

In \exists Intro and \forall Elim there are no restrictions, and the term t can be anything, so we do not have to worry about any conditions. On the other hand, in the \exists Elim and \forall Intro rules, the eigenvariable condition requires that the constant symbol a does not occur anywhere in the conclusion or in an undischarged assumption. The condition is necessary to ensure that the system is sound, i.e., only derives sentences from undischarged assumptions from which they follow. Without this condition, the following would be allowed:

$$\frac{\exists x \varphi(x) \quad \frac{[\varphi(a)]^1}{\forall x \varphi(x)} * \forall \text{Intro}}{\forall x \varphi(x)} \exists \text{Elim}$$

However, $\exists x \varphi(x) \not\equiv \forall x \varphi(x)$.

As the elimination rules for quantifiers only allow substituting closed terms for variables, it follows that any formula that can be derived from a set of sentences is itself a sentence.

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Bibliography