ntd.1 Derivability and the Propositional Connectives

fol:ntd:ppr: sec

We establish that the derivability relation $\vdash$ of natural deduction is strong enough to establish some basic facts involving the propositional connectives, such as that $\varphi \land \psi \vdash \varphi$ and $\varphi, \varphi \to \psi \vdash \psi$ (modus ponens). These facts are needed for the proof of the completeness theorem. explanation

fol:ntd:ppr: prop:provability-land

Proposition ntd.1.

fol:ntd:ppr: prop:provability-land-left

1. *Both* $\varphi \land \psi \vdash \varphi$ *and* $\varphi \land \psi \vdash \psi$

fol:ntd:ppr: prop:provability-land-right

2. $\varphi, \psi \vdash \varphi \land \psi$.

Proof. 1. We can derive both

$$\frac{\varphi \land \psi}{\varphi}\,\land\text{Elim} \qquad \frac{\varphi \land \psi}{\psi}\,\land\text{Elim}$$

2. We can derive:

$$\frac{\varphi \qquad \psi}{\varphi \land \psi}\,\land\text{Intro}$$

□

fol:ntd:ppr: prop:provability-lor

Proposition ntd.2.

1. $\varphi \lor \psi, \neg\varphi, \neg\psi$ *is inconsistent.*
2. *Both* $\varphi \vdash \varphi \lor \psi$ *and* $\psi \vdash \varphi \lor \psi$.

Proof. 1. Consider the following derivation:

$$\frac{\varphi \lor \psi \qquad \dfrac{\neg\varphi \qquad [\varphi]^1}{\bot}\,\neg\text{Elim} \qquad \dfrac{\neg\psi \qquad [\psi]^1}{\bot}\,\neg\text{Elim}}{\bot}\,{}_1\,\lor\text{Elim}$$

This is a derivation of $\bot$ from undischarged assumptions $\varphi \lor \psi$, $\neg\varphi$, and $\neg\psi$.

2. We can derive both

$$\frac{\varphi}{\varphi \lor \psi}\,\lor\text{Intro} \qquad \frac{\psi}{\varphi \lor \psi}\,\lor\text{Intro}$$

□

fol:ntd:ppr: prop:provability-lif

Proposition ntd.3.

fol:ntd:ppr: prop:provability-lif-left

1. $\varphi, \varphi \to \psi \vdash \psi$.

fol:ntd:ppr: prop:provability-lif-right

2. *Both* $\neg\varphi \vdash \varphi \to \psi$ *and* $\psi \vdash \varphi \to \psi$.

Proof. 1. We can derive:

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \to\text{Elim}$$

2. This is shown by the following two derivations:

$$\dfrac{\dfrac{\dfrac{\neg\varphi \qquad [\varphi]^1}{\bot} \neg\text{Elim}}{\psi} \bot_I}{\varphi \to \psi} {}^{1}\to\text{Intro} \qquad\qquad \dfrac{\psi}{\varphi \to \psi} \to\text{Intro}$$

Note that $\to$Intro may, but does not have to, discharge the assumption φ.
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Bibliography