Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

Proposition ntd.1. If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

Proof. Let the derivation of $\varphi$ from $\Gamma$ be $\delta_1$ and the derivation of $\bot$ from $\Gamma \cup \{\varphi\}$ be $\delta_2$. We can then derive:

\[
\begin{array}{c}
\Gamma, [\varphi]^1 \\
\vdots \\
\delta_2 \\
\vdots \\
\bot \\
\hline
\varphi \\
\hline
\Gamma, [\neg \varphi]^1 \\
\vdots \\
\delta_1 \\
\vdots \\
\bot \\
\hline
\end{array}
\]

In the new derivation, the assumption $\varphi$ is discharged, so it is a derivation from $\Gamma$.

Proposition ntd.2. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

Proof. First suppose $\Gamma \vdash \varphi$, i.e., there is a derivation $\delta_0$ of $\varphi$ from undischarged assumptions $\Gamma$. We obtain a derivation of $\bot$ from $\Gamma \cup \{\neg \varphi\}$ as follows:

\[
\begin{array}{c}
\Gamma \\
\vdots \\
\delta_0 \\
\vdots \\
\neg \varphi \\
\hline
\varphi \\
\hline
\bot \\
\hline
\end{array}
\]

Now assume $\Gamma \cup \{\neg \varphi\}$ is inconsistent, and let $\delta_1$ be the corresponding derivation of $\bot$ from undischarged assumptions in $\Gamma \cup \{\neg \varphi\}$. We obtain a derivation of $\varphi$ from $\Gamma$ alone by using $\bot_C$:

\[
\begin{array}{c}
\Gamma, [\neg \varphi]^1 \\
\vdots \\
\delta_1 \\
\vdots \\
\bot \\
\hline
\varphi \\
\hline
\bot_C \\
\end{array}
\]

Problem ntd.1. Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

Proposition ntd.3. If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.
Proof. Suppose $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$. Then there is a derivation $\delta$ of $\varphi$ from $\Gamma$. Consider this simple application of the $\neg$-Elim rule:

$$
\begin{array}{c}
\Gamma \\
\vdots \\
\delta \\
\neg \varphi \\
\varphi \\
\end{array}
\quad \neg\text{Elim}
$$

Since $\neg \varphi \in \Gamma$, all undischarged assumptions are in $\Gamma$, this shows that $\Gamma \vdash \bot$. \hfill $\square$

**Proposition ntd.4.** If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

Proof. There are derivations $\delta_1$ and $\delta_2$ of $\bot$ from $\Gamma \cup \{\varphi\}$ and $\bot$ from $\Gamma \cup \{\neg \varphi\}$, respectively. We can then derive

$$
\begin{array}{c}
\Gamma, [\neg \varphi]^2 \\
\vdots \\
\delta_2 \\
\bot \\
\end{array}
\quad
\begin{array}{c}
\Gamma, [\varphi]^1 \\
\vdots \\
\delta_1 \\
\bot \\
\end{array}
\quad
\begin{array}{c}
\bot \\
\neg\text{Intro} \\
\neg\text{Intro} \\
\neg\text{Intro} \\
\neg\text{Intro} \\
\neg\text{Elim} \\
\end{array}
\quad
\begin{array}{c}
\bot \\
\end{array}
$$

Since the assumptions $\varphi$ and $\neg \varphi$ are discharged, this is a derivation of $\bot$ from $\Gamma$ alone. Hence $\Gamma$ is inconsistent. \hfill $\square$

**Photo Credits**

**Bibliography**