

ntd.1 Proof-Theoretic Notions

fol:ntd:ptn:
sec

This section collects the definitions the provability relation and consistency for natural deduction.

Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain **sentences** from others. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

explanation

Definition ntd.1 (Theorems). A **sentence** φ is a *theorem* if there is a **derivation** of φ in natural deduction in which all assumptions are **discharged**. We write $\vdash \varphi$ if φ is a theorem and $\not\vdash \varphi$ if it is not.

Definition ntd.2 (Derivability). A **sentence** φ is *derivable* from a set of **sentences** Γ , $\Gamma \vdash \varphi$, if there is a **derivation** with conclusion φ and in which every assumption is either **discharged** or is in Γ . If φ is not **derivable** from Γ we write $\Gamma \not\vdash \varphi$.

Definition ntd.3 (Consistency). A set of **sentences** Γ is *inconsistent* iff $\Gamma \vdash \perp$. If Γ is not inconsistent, i.e., if $\Gamma \not\vdash \perp$, we say it is *consistent*.

fol:ntd:ptn:
prop:reflexivity

Proposition ntd.4 (Reflexivity). If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

Proof. The assumption φ by itself is a **derivation** of φ where every **undischarged** assumption (i.e., φ) is in Γ . \square

fol:ntd:ptn:
prop:monotonicity

Proposition ntd.5 (Monotonicity). If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.

Proof. Any **derivation** of φ from Γ is also a **derivation** of φ from Δ . \square

fol:ntd:ptn:
prop:transitivity

Proposition ntd.6 (Transitivity). If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.

Proof. If $\Gamma \vdash \varphi$, there is a **derivation** δ_0 of φ with all **undischarged** assumptions in Γ . If $\{\varphi\} \cup \Delta \vdash \psi$, then there is a **derivation** δ_1 of ψ with all **undischarged** assumptions in $\{\varphi\} \cup \Delta$. Now consider:

$$\begin{array}{c}
 \Delta, [\varphi]^1 \\
 \vdots \\
 \delta_1 \\
 \vdots \\
 \psi \\
 \hline
 \varphi \rightarrow \psi \quad \rightarrow \text{Intro} \\
 \hline
 \psi
 \end{array}
 \quad
 \begin{array}{c}
 \Gamma \\
 \vdots \\
 \delta_0 \\
 \vdots \\
 \varphi \\
 \hline
 \psi \quad \rightarrow \text{Elim}
 \end{array}$$

The **undischarged** assumptions are now all among $\Gamma \cup \Delta$, so this shows $\Gamma \cup \Delta \vdash \psi$. \square

When $\Gamma = \{\varphi_1, \varphi_2, \dots, \varphi_k\}$ is a finite set we may use the simplified notation $\varphi_1, \varphi_2, \dots, \varphi_k \vdash \psi$ for $\Gamma \vdash \psi$, in particular $\varphi \vdash \psi$ means that $\{\varphi\} \vdash \psi$.

Note that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \dots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i , then $\Gamma \vdash \psi$.

Proposition ntd.7. *The following are equivalent.*

*fol.ntd.ptn:
prop:incons*

1. Γ is inconsistent.
2. $\Gamma \vdash \varphi$ for every **sentence** φ .
3. $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ for some **sentence** φ .

Proof. Exercise. \square

Problem ntd.1. Prove **Proposition ntd.7**

Proposition ntd.8 (Compactness).

*fol.ntd.ptn:
prop:proves-compact*

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.
2. If every finite subset of Γ is consistent, then Γ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a **derivation** δ of φ from Γ . Let Γ_0 be the set of **undischarged** assumptions of δ . Since any **derivation** is finite, Γ_0 can only contain finitely many **sentences**. So, δ is a **derivation** of φ from a finite $\Gamma_0 \subseteq \Gamma$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \perp$. \square

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Bibliography