

ntd.1 Derivations with Identity predicate

fol:ntd:ide: Derivations with identity predicate require additional inference rules.
sec

$$\frac{}{t = t} =\text{Intro} \qquad \frac{t_1 = t_2 \quad \varphi(t_1)}{\varphi(t_2)} =\text{Elim}$$

$$\frac{t_1 = t_2 \quad \varphi(t_2)}{\varphi(t_1)} =\text{Elim}$$

In the above rules, t , t_1 , and t_2 are closed terms. The =Intro rule allows us to **derive** any identity statement of the form $t = t$ outright, from no assumptions.

Example ntd.1. If s and t are closed terms, then $\varphi(s), s = t \vdash \varphi(t)$:

$$\frac{s = t \quad \varphi(s)}{\varphi(t)} =\text{Elim}$$

This may be familiar as the “principle of substitutability of identicals,” or Leibniz’ Law.

Problem ntd.1. Prove that = is both symmetric and transitive, i.e., give **derivations** of $\forall x \forall y (x = y \rightarrow y = x)$ and $\forall x \forall y \forall z ((x = y \wedge y = z) \rightarrow x = z)$

Example ntd.2. We **derive** the **sentence**

$$\forall x \forall y ((\varphi(x) \wedge \varphi(y)) \rightarrow x = y)$$

from the **sentence**

$$\exists x \forall y (\varphi(y) \rightarrow y = x)$$

We develop the **derivation** backwards:

$$\begin{array}{c} \exists x \forall y (\varphi(y) \rightarrow y = x) \quad [\varphi(a) \wedge \varphi(b)]^1 \\ \vdots \\ a = b \\ \frac{}{((\varphi(a) \wedge \varphi(b)) \rightarrow a = b)} \rightarrow\text{Intro} \\ \frac{}{\forall y ((\varphi(a) \wedge \varphi(y)) \rightarrow a = y)} \forall\text{Intro} \\ \frac{}{\forall x \forall y ((\varphi(x) \wedge \varphi(y)) \rightarrow x = y)} \forall\text{Intro} \end{array}$$

We’ll now have to use the main assumption: since it is an existential **formula**, we use $\exists\text{Elim}$ to **derive** the intermediary conclusion $a = b$.

$$\begin{array}{c}
[\forall y (\varphi(y) \rightarrow y = c)]^2 \\
[\varphi(a) \wedge \varphi(b)]^1 \\
\vdots \\
a = b \\
\hline
\frac{\exists x \forall y (\varphi(y) \rightarrow y = x) \quad a = b}{\exists \text{Elim}} \\
\hline
\frac{1 \quad \frac{a = b}{((\varphi(a) \wedge \varphi(b)) \rightarrow a = b)} \rightarrow \text{Intro}}{\forall y ((\varphi(a) \wedge \varphi(y)) \rightarrow a = y)} \forall \text{Intro} \\
\hline
\frac{\forall y ((\varphi(a) \wedge \varphi(y)) \rightarrow a = y)}{\forall x \forall y ((\varphi(x) \wedge \varphi(y)) \rightarrow x = y)} \forall \text{Intro}
\end{array}$$

The sub-derivation on the top right is completed by using its assumptions to show that $a = c$ and $b = c$. This requires two separate derivations. The derivation for $a = c$ is as follows:

$$\frac{\frac{[\forall y (\varphi(y) \rightarrow y = c)]^2}{\varphi(a) \rightarrow a = c} \forall \text{Elim} \quad \frac{[\varphi(a) \wedge \varphi(b)]^1}{\varphi(a)} \wedge \text{Elim}}{a = c} \rightarrow \text{Elim}$$

From $a = c$ and $b = c$ we derive $a = b$ by =Elim.

Problem ntd.2. Give derivations of the following formulas:

1. $\forall x \forall y ((x = y \wedge \varphi(x)) \rightarrow \varphi(y))$
2. $\exists x \varphi(x) \wedge \forall y \forall z ((\varphi(y) \wedge \varphi(z)) \rightarrow y = z) \rightarrow \exists x (\varphi(x) \wedge \forall y (\varphi(y) \rightarrow y = x))$

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Bibliography