

ntd.1 Derivations

fol:ntd:der: We've said what an assumption is, and we've given the rules of inference. explanation
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Derivations in natural deduction are inductively generated from these: each **derivation** either is an assumption on its own, or consists of one, two, or three **derivations** followed by a correct inference.

Definition ntd.1 (Derivation). A *derivation* of a sentence φ from assumptions Γ is a tree of **sentences** satisfying the following conditions:

1. The topmost **sentences** of the tree are either in Γ or are **discharged** by an inference in the tree.
2. The bottommost **sentence** of the tree is φ .
3. Every **sentence** in the tree except the sentence φ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that **sentence** in the tree.

We then say that φ is the *conclusion* of the **derivation** and that φ is *derivable* from Γ .

Example ntd.2. Every assumption on its own is a **derivation**. So, e.g., χ by itself is a **derivation**, and so is θ by itself. We can obtain a new **derivation** from these by applying, say, the \wedge Intro rule,

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any **sentences**, e.g., by χ and θ . Then the conclusion would be $\chi \wedge \theta$, and so

$$\frac{\chi \quad \theta}{\chi \wedge \theta} \wedge\text{Intro}$$

is a correct **derivation**. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ . Thus,

$$\frac{\theta \quad \chi}{\theta \wedge \chi} \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, \rightarrow Intro, which allows us to conclude a conditional and allows us to **discharge** any assumption that is identical to the antecedent of that conditional. So both of the following would be correct **derivations**:

$$1 \frac{\frac{[\chi]^1}{\chi \wedge \theta} \wedge\text{Intro}}{\chi \rightarrow (\chi \wedge \theta)} \rightarrow\text{Intro} \quad 1 \frac{\frac{\chi \quad [\theta]^1}{\chi \wedge \theta} \wedge\text{Intro}}{\theta \rightarrow (\chi \wedge \theta)} \rightarrow\text{Intro}$$

Remember that discharging of assumptions is a permission, not a requirement: we don't have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption φ to be **discharged**:

$$1 \frac{\psi}{\varphi \rightarrow \psi} \rightarrow \text{Intro}$$

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Bibliography