

ntd.1 Derivations

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sec We've said what an assumption is, and we've given the rules of inference. explanation
Derivations in natural deduction are inductively generated from these: each **derivation** either is an assumption on its own, or consists of one, two, or three **derivations** followed by a correct inference.

Definition ntd.1 (Derivation). A *derivation* of a sentence φ from assumptions Γ is a finite tree of **sentences** satisfying the following conditions:

1. The topmost **sentences** of the tree are either in Γ or are **discharged** by an inference in the tree.
2. The bottommost **sentence** of the tree is φ .
3. Every **sentence** in the tree except the sentence φ at the bottom is a premise of a correct application of an inference rule whose conclusion stands directly below that **sentence** in the tree.

We then say that φ is the *conclusion* of the **derivation** and Γ its **undischarged** assumptions.

If a **derivation** of φ from Γ exists, we say that φ is *derivable* from Γ , or in symbols: $\Gamma \vdash \varphi$. If there is a **derivation** of φ in which every assumption is **discharged**, we write $\vdash \varphi$.

Example ntd.2. Every assumption on its own is a **derivation**. So, e.g., φ by itself is a **derivation**, and so is ψ by itself. We can obtain a new **derivation** from these by applying, say, the \wedge Intro rule,

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro}$$

These rules are meant to be general: we can replace the φ and ψ in it with any **sentences**, e.g., by χ and θ . Then the conclusion would be $\chi \wedge \theta$, and so

$$\frac{\chi \quad \theta}{\chi \wedge \theta} \wedge\text{Intro}$$

is a correct **derivation**. Of course, we can also switch the assumptions, so that θ plays the role of φ and χ that of ψ . Thus,

$$\frac{\theta \quad \chi}{\theta \wedge \chi} \wedge\text{Intro}$$

is also a correct derivation.

We can now apply another rule, say, \rightarrow Intro, which allows us to conclude a conditional and allows us to **discharge** any assumption that is identical to the antecedent of that conditional. So both of the following would be correct **derivations**:

$$\frac{\frac{[\chi]^1 \quad \theta}{\chi \wedge \theta} \wedge \text{Intro}}{\chi \rightarrow (\chi \wedge \theta)} \rightarrow \text{Intro} \quad \frac{\frac{\chi \quad [\theta]^1}{\chi \wedge \theta} \wedge \text{Intro}}{\theta \rightarrow (\chi \wedge \theta)} \rightarrow \text{Intro}$$

They show, respectively, that $\theta \vdash \chi \rightarrow (\chi \wedge \theta)$ and $\chi \vdash \theta \rightarrow (\chi \wedge \theta)$.

Remember that discharging of assumptions is a permission, not a requirement: we don't have to discharge the assumptions. In particular, we can apply a rule even if the assumptions are not present in the derivation. For instance, the following is legal, even though there is no assumption φ to be discharged:

$$\frac{\psi}{\varphi \rightarrow \psi} \rightarrow \text{Intro}$$

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Bibliography