

mat.1 The Theory of Sets

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Almost all of mathematics can be developed in the theory of sets. Developing mathematics in this theory involves a number of things. First, it requires a set of axioms for the relation \in . A number of different axiom systems have been developed, sometimes with conflicting properties of \in . The axiom system known as **ZFC**, Zermelo–Fraenkel set theory with the axiom of choice stands out: it is by far the most widely used and studied, because it turns out that its axioms suffice to prove almost all the things mathematicians expect to be able to prove. But before that can be established, it first is necessary to make clear how we can even *express* all the things mathematicians would like to express. For starters, the language contains no **constant symbols** or **function symbols**, so it seems at first glance unclear that we can talk about particular sets (such as \emptyset or \mathbb{N}), can talk about operations on sets (such as $X \cup Y$ and $\wp(X)$), let alone other constructions which involve things other than sets, such as relations and functions.

To begin with, “is an element of” is not the only relation we are interested in: “is a subset of” seems almost as important. But we can *define* “is a subset of” in terms of “is an element of.” To do this, we have to find a **formula** $\varphi(x, y)$ in the language of set theory which is satisfied by a pair of sets $\langle X, Y \rangle$ iff $X \subseteq Y$. But X is a subset of Y just in case all **elements** of X are also **elements** of Y . So we can define \subseteq by the formula

$$\forall z (z \in x \rightarrow z \in y)$$

Now, whenever we want to use the relation \subseteq in a formula, we could instead use that formula (with x and y suitably replaced, and the bound variable z renamed if necessary). For instance, extensionality of sets means that if any sets x and y are contained in each other, then x and y must be the same set. This can be expressed by $\forall x \forall y ((x \subseteq y \wedge y \subseteq x) \rightarrow x = y)$, or, if we replace \subseteq by the above definition, by

$$\forall x \forall y ((\forall z (z \in x \rightarrow z \in y) \wedge \forall z (z \in y \rightarrow z \in x)) \rightarrow x = y).$$

This is in fact one of the axioms of **ZFC**, the “axiom of extensionality.”

There is no **constant symbol** for \emptyset , but we can express “ x is empty” by $\neg \exists y y \in x$. Then “ \emptyset exists” becomes the **sentence** $\exists x \neg \exists y y \in x$. This is another axiom of **ZFC**. (Note that the axiom of extensionality implies that there is only one empty set.) Whenever we want to talk about \emptyset in the language of set theory, we would write this as “there is a set that’s empty and . . .” As an example, to express the fact that \emptyset is a subset of every set, we could write

$$\exists x (\neg \exists y y \in x \wedge \forall z x \subseteq z)$$

where, of course, $x \subseteq z$ would in turn have to be replaced by its definition.

To talk about operations on sets, such as $X \cup Y$ and $\wp(X)$, we have to use a similar trick. There are no function symbols in the language of set theory,

but we can express the functional relations $X \cup Y = Z$ and $\wp(X) = Y$ by

$$\begin{aligned} \forall u ((u \in x \vee u \in y) \leftrightarrow u \in z) \\ \forall u (u \subseteq x \leftrightarrow u \in y) \end{aligned}$$

since the **elements** of $X \cup Y$ are exactly the sets that are either **elements** of X or **elements** of Y , and the **elements** of $\wp(X)$ are exactly the subsets of X . However, this doesn't allow us to use $x \cup y$ or $\wp(x)$ as if they were terms: we can only use the entire **formulas** that define the relations $X \cup Y = Z$ and $\wp(X) = Y$. In fact, we do not know that these relations are ever satisfied, i.e., we do not know that unions and power sets always exist. For instance, the **sentence** $\forall x \exists y \wp(x) = y$ is another axiom of **ZFC** (the power set axiom).

Now what about talk of ordered pairs or functions? Here we have to explain how we can think of ordered pairs and functions as special kinds of sets. One way to define the ordered pair $\langle x, y \rangle$ is as the set $\{\{x\}, \{x, y\}\}$. But like before, we cannot introduce a **function symbol** that names this set; we can only define the relation $\langle x, y \rangle = z$, i.e., $\{\{x\}, \{x, y\}\} = z$:

$$\forall u (u \in z \leftrightarrow (\forall v (v \in u \leftrightarrow v = x) \vee \forall v (v \in u \leftrightarrow (v = x \vee v = y))))$$

This says that the **elements** u of z are exactly those sets which either have x as its only **element** or have x and y as its only **elements** (in other words, those sets that are either identical to $\{x\}$ or identical to $\{x, y\}$). Once we have this, we can say further things, e.g., that $X \times Y = Z$:

$$\forall z (z \in Z \leftrightarrow \exists x \exists y (x \in X \wedge y \in Y \wedge \langle x, y \rangle = z))$$

A function $f: X \rightarrow Y$ can be thought of as the relation $f(x) = y$, i.e., as the set of pairs $\{\langle x, y \rangle : f(x) = y\}$. We can then say that a set f is a function from X to Y if (a) it is a relation $\subseteq X \times Y$, (b) it is total, i.e., for all $x \in X$ there is some $y \in Y$ such that $\langle x, y \rangle \in f$ and (c) it is functional, i.e., whenever $\langle x, y \rangle, \langle x, y' \rangle \in f$, $y = y'$ (because values of functions must be unique). So “ f is a function from X to Y ” can be written as:

$$\begin{aligned} \forall u (u \in f \rightarrow \exists x \exists y (x \in X \wedge y \in Y \wedge \langle x, y \rangle = u)) \wedge \\ \forall x (x \in X \rightarrow (\exists y (y \in Y \wedge \text{maps}(f, x, y)) \wedge \\ (\forall y \forall y' ((\text{maps}(f, x, y) \wedge \text{maps}(f, x, y')) \rightarrow y = y')))) \end{aligned}$$

where $\text{maps}(f, x, y)$ abbreviates $\exists v (v \in f \wedge \langle x, y \rangle = v)$ (this **formula** expresses “ $f(x) = y$ ”).

It is now also not hard to express that $f: X \rightarrow Y$ is **injective**, for instance:

$$\begin{aligned} f: X \rightarrow Y \wedge \forall x \forall x' ((x \in X \wedge x' \in X \wedge \\ \exists y (\text{maps}(f, x, y) \wedge \text{maps}(f, x', y))) \rightarrow x = x') \end{aligned}$$

A function $f: X \rightarrow Y$ is **injective** iff, whenever f maps $x, x' \in X$ to a single y , $x = x'$. If we abbreviate this formula as $\text{inj}(f, X, Y)$, we're already in a position

to state in the language of set theory something as non-trivial as Cantor’s theorem: there is no **injective** function from $\wp(X)$ to X :

$$\forall X \forall Y (\wp(X) = Y \rightarrow \neg \exists f \text{ inj}(f, Y, X))$$

One might think that set theory requires another axiom that guarantees the existence of a set for every defining property. If $\varphi(x)$ is a formula of set theory with the variable x free, we can consider the **sentence**

$$\exists y \forall x (x \in y \leftrightarrow \varphi(x)).$$

This **sentence** states that there is a set y whose **elements** are all and only those x that satisfy $\varphi(x)$. This schema is called the “comprehension principle.” It looks very useful; unfortunately it is inconsistent. Take $\varphi(x) \equiv \neg x \in x$, then the comprehension principle states

$$\exists y \forall x (x \in y \leftrightarrow x \notin x),$$

i.e., it states the existence of a set of all sets that are not **elements** of themselves. No such set can exist—this is Russell’s Paradox. **ZFC**, in fact, contains a restricted—and consistent—version of this principle, the separation principle:

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \varphi(x))).$$

Problem mat.1. Show that the comprehension principle is inconsistent by giving a **derivation** that shows

$$\exists y \forall x (x \in y \leftrightarrow x \notin x) \vdash \perp.$$

It may help to first show $(A \rightarrow \neg A) \wedge (\neg A \rightarrow A) \vdash \perp$.

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Bibliography