

## mat.1 Expressing Relations in a Structure

fol:mat:exr: One main use **formulas** can be put to is to express properties and relations in **a structure**  $\mathfrak{M}$  in terms of the primitives of the language  $\mathcal{L}$  of  $\mathfrak{M}$ . By this we mean the following: the **domain** of  $\mathfrak{M}$  is a set of objects. The **constant symbols**, **function symbols**, and **predicate symbols** are interpreted in  $\mathfrak{M}$  by some objects in  $|\mathfrak{M}|$ , functions on  $|\mathfrak{M}|$ , and relations on  $|\mathfrak{M}|$ . For instance, if  $A_0^2$  is in  $\mathcal{L}$ , then  $\mathfrak{M}$  assigns to it a relation  $R = A_0^{2\mathfrak{M}}$ . Then the formula  $A_0^2(v_1, v_2)$  *expresses* that very relation, in the following sense: if a variable assignment  $s$  maps  $v_1$  to  $a \in |\mathfrak{M}|$  and  $v_2$  to  $b \in |\mathfrak{M}|$ , then

$$Rab \text{ iff } \mathfrak{M}, s \models A_0^2(v_1, v_2).$$

Note that we have to involve variable assignments here: we can't just say " $Rab$  iff  $\mathfrak{M} \models A_0^2(a, b)$ " because  $a$  and  $b$  are not symbols of our language: they are **elements** of  $|\mathfrak{M}|$ .

Since we don't just have atomic **formulas**, but can combine them using the logical connectives and the quantifiers, more complex **formulas** can define other relations which aren't directly built into  $\mathfrak{M}$ . We're interested in how to do that, and specifically, which relations we can define in **a structure**.

**Definition mat.1.** Let  $\varphi(v_1, \dots, v_n)$  be a **formula** of  $\mathcal{L}$  in which only  $v_1, \dots, v_n$  occur free, and let  $\mathfrak{M}$  be **a structure** for  $\mathcal{L}$ .  $\varphi(v_1, \dots, v_n)$  *expresses the relation*  $R \subseteq |\mathfrak{M}|^n$  iff

$$Ra_1 \dots a_n \text{ iff } \mathfrak{M}, s \models \varphi(v_1, \dots, v_n)$$

for any variable assignment  $s$  with  $s(v_i) = a_i$  ( $i = 1, \dots, n$ ).

**Example mat.2.** In the standard model of arithmetic  $\mathfrak{N}$ , the **formula**  $v_1 < v_2 \vee v_1 = v_2$  expresses the  $\leq$  relation on  $\mathbb{N}$ . The **formula**  $v_2 = v_1'$  expresses the successor relation, i.e., the relation  $R \subseteq \mathbb{N}^2$  where  $Rnm$  holds if  $m$  is the successor of  $n$ . The formula  $v_1 = v_2'$  expresses the predecessor relation. The **formulas**  $\exists v_3 (v_3 \neq 0 \wedge v_2 = (v_1 + v_3))$  and  $\exists v_3 (v_1 + v_3' = v_2)$  both express the  $<$  relation. This means that the predicate symbol  $<$  is actually superfluous in the language of arithmetic; it can be defined.

This idea is not just interesting in specific **structures**, but generally whenever we use a language to describe an intended model or models, i.e., when we consider theories. These theories often only contain a few **predicate symbols** as basic symbols, but in the domain they are used to describe often many other relations play an important role. If these other relations can be systematically expressed by the relations that interpret the basic **predicate symbols** of the language, we say we can *define* them in the language.

**Problem mat.1.** Find **formulas** in  $\mathcal{L}_A$  which define the following relations:

1.  $n$  is between  $i$  and  $j$ ;

2.  $n$  evenly divides  $m$  (i.e.,  $m$  is a multiple of  $n$ );
3.  $n$  is a prime number (i.e., no number other than 1 and  $n$  evenly divides  $n$ ).

**Problem mat.2.** Suppose the formula  $\varphi(v_1, v_2)$  expresses the relation  $R \subseteq |\mathfrak{M}|^2$  in a structure  $\mathfrak{M}$ . Find formulas that express the following relations:

1. the inverse  $R^{-1}$  of  $R$ ;
2. the relative product  $R \mid R$ ;

Can you find a way to express  $R^+$ , the transitive closure of  $R$ ?

**Problem mat.3.** Let  $\mathcal{L}$  be the language containing a 2-place predicate symbol  $<$  only (no other constant symbols, function symbols or predicate symbols—except of course  $=$ ). Let  $\mathfrak{N}$  be the structure such that  $|\mathfrak{N}| = \mathbb{N}$ , and  $<^{\mathfrak{N}} = \{\langle n, m \rangle : n < m\}$ . Prove the following:

1.  $\{0\}$  is definable in  $\mathfrak{N}$ ;
2.  $\{1\}$  is definable in  $\mathfrak{N}$ ;
3.  $\{2\}$  is definable in  $\mathfrak{N}$ ;
4. for each  $n \in \mathbb{N}$ , the set  $\{n\}$  is definable in  $\mathfrak{N}$ ;
5. every finite subset of  $|\mathfrak{N}|$  is definable in  $\mathfrak{N}$ ;
6. every co-finite subset of  $|\mathfrak{N}|$  is definable in  $\mathfrak{N}$  (where  $X \subseteq \mathbb{N}$  is co-finite iff  $\mathbb{N} \setminus X$  is finite).

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## Bibliography