

## com.1 Lindenbaum's Lemma

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sec We now prove a lemma that shows that any consistent set of **sentences** is contained in some set of sentences which is not just consistent, but also **complete**. explanation  
The proof works by adding one **sentence** at a time, guaranteeing at each step that the set remains consistent. We do this so that for every  $\varphi$ , either  $\varphi$  or  $\neg\varphi$  gets added at some stage. The union of all stages in that construction then contains either  $\varphi$  or its negation  $\neg\varphi$  and is thus complete. It is also consistent, since we made sure at each stage not to introduce an inconsistency.

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lem:lindenbaum **Lemma com.1 (Lindenbaum's Lemma).** *Every consistent set  $\Gamma$  in a language  $\mathcal{L}$  can be extended to a **complete** and consistent set  $\Gamma^*$ .*

*Proof.* Let  $\Gamma$  be consistent. Let  $\varphi_0, \varphi_1, \dots$  be an enumeration of all the **sentences** of  $\mathcal{L}$ . Define  $\Gamma_0 = \Gamma$ , and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

Let  $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$ .

Each  $\Gamma_n$  is consistent:  $\Gamma_0$  is consistent by definition. If  $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$ , this is because the latter is consistent. If it isn't,  $\Gamma_{n+1} = \Gamma_n \cup \{\neg\varphi_n\}$ . We have to verify that  $\Gamma_n \cup \{\neg\varphi_n\}$  is consistent. Suppose it's not. Then *both*  $\Gamma_n \cup \{\varphi_n\}$  and  $\Gamma_n \cup \{\neg\varphi_n\}$  are inconsistent. This means that  $\Gamma_n$  would be inconsistent by **????????????????**, contrary to the induction hypothesis.

For every  $n$  and every  $i < n$ ,  $\Gamma_i \subseteq \Gamma_n$ . This follows by a simple induction on  $n$ . For  $n = 0$ , there are no  $i < 0$ , so the claim holds automatically. For the inductive step, suppose it is true for  $n$ . We have  $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_n\}$  or  $= \Gamma_n \cup \{\neg\varphi_n\}$  by construction. So  $\Gamma_n \subseteq \Gamma_{n+1}$ . If  $i < n$ , then  $\Gamma_i \subseteq \Gamma_n$  by inductive hypothesis, and so  $\subseteq \Gamma_{n+1}$  by transitivity of  $\subseteq$ .

From this it follows that every finite subset of  $\Gamma^*$  is a subset of  $\Gamma_n$  for some  $n$ , since each  $\psi \in \Gamma^*$  not already in  $\Gamma_0$  is added at some stage  $i$ . If  $n$  is the last one of these, then all  $\psi$  in the finite subset are in  $\Gamma_n$ . So, every finite subset of  $\Gamma^*$  is consistent. By **????????????????**,  $\Gamma^*$  is consistent.

Every **sentence** of  $\text{Frm}(\mathcal{L})$  appears on the list used to define  $\Gamma^*$ . If  $\varphi_n \notin \Gamma^*$ , then that is because  $\Gamma_n \cup \{\varphi_n\}$  was inconsistent. But then  $\neg\varphi_n \in \Gamma^*$ , so  $\Gamma^*$  is **complete**.  $\square$

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## Bibliography