com.1 Identity

fol:com:ide:sec

The construction of the term model given in the preceding section is enough to establish completeness for first-order logic for sets Γ that do not contain $=$. The term model satisfies every $\varphi \in \Gamma^*$ which does not contain $=$ (and hence all $\varphi \in \Gamma$). It does not work, however, if $=$ is present. The reason is that Γ^* then may contain a sentence $t = t'$, but in the term model the value of any term is that term itself. Hence, if t and t' are different terms, their values in the term model—i.e., t and t', respectively—are different, and so $t = t'$ is false. We can fix this, however, using a construction known as "factoring." explanation

Definition com.1. Let Γ^* be a consistent and complete set of sentences in $\mathcal{L}$. We define the relation $\approx$ on the set of closed terms of $\mathcal{L}$ by

$$t \approx t' \quad \text{iff} \quad t = t' \in \Gamma^*$$

fol:com:ide:prop:approx-equiv

Proposition com.2. *The relation $\approx$ has the following properties:*

1. *$\approx$ is reflexive.*
2. *$\approx$ is symmetric.*
3. *$\approx$ is transitive.*
4. *If $t \approx t'$, f is a function symbol, and t_1, ..., t_{i-1}, t_{i+1}, ..., t_n are closed terms, then*

$$f(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \approx f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n).$$

5. *If $t \approx t'$, R is a predicate symbol, and t_1, ..., t_{i-1}, t_{i+1}, ..., t_n are closed terms, then*

$$R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \in \Gamma^* \text{ iff } R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n) \in \Gamma^*.$$

Proof. Since Γ^* is consistent and complete, $t = t' \in \Gamma^*$ iff $\Gamma^* \vdash t = t'$. Thus it is enough to show the following:

1. $\Gamma^* \vdash t = t$ for all closed terms t.
2. If $\Gamma^* \vdash t = t'$ then $\Gamma^* \vdash t' = t$.
3. If $\Gamma^* \vdash t = t'$ and $\Gamma^* \vdash t' = t''$, then $\Gamma^* \vdash t = t''$.
4. If $\Gamma^* \vdash t = t'$, then

$$\Gamma^* \vdash f(t_1, \dots, t_{i-1}, t, t_{i+1}, , \dots, t_n) = f(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$$

for every n-place function symbol f and closed terms t_1, ..., t_{i-1}, t_{i+1}, ..., t_n.

5. If $\Gamma^* \vdash t = t'$ and $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n)$, then $\Gamma^* \vdash R(t_1, \dots, t_{i-1}, t', t_{i+1}, \dots, t_n)$ for every n-place predicate symbol R and closed terms t_1, …, t_{i-1}, t_{i+1}, …, t_n. □

Problem com.1. Complete the proof of Proposition com.2.

Definition com.3. Suppose Γ^* is a consistent and complete set in a language $\mathcal{L}$, t is a closed term, and $\approx$ as in the previous definition. Then:

$$[t]_\approx = \{t' : t' \in \mathrm{Trm}(\mathcal{L}), t \approx t'\}$$

and $\mathrm{Trm}(\mathcal{L})/_\approx = \{[t]_\approx : t \in \mathrm{Trm}(\mathcal{L})\}$.

Definition com.4. Let $\mathfrak{M} = \mathfrak{M}(\Gamma^*)$ be the term model for Γ^* from **??**. Then $\mathfrak{M}/_\approx$ is the following structure: fol:com:ide: defn:term-model-factor

1. $|\mathfrak{M}/_\approx| = \mathrm{Trm}(\mathcal{L})/_\approx$.
2. $c^{\mathfrak{M}/_\approx} = [c]_\approx$
3. $f^{\mathfrak{M}/_\approx}([t_1]_\approx, \dots, [t_n]_\approx) = [f(t_1, \dots, t_n)]_\approx$
4. $\langle [t_1]_\approx, \dots, [t_n]_\approx \rangle \in R^{\mathfrak{M}/_\approx}$ iff $\mathfrak{M} \vDash R(t_1, \dots, t_n)$, i.e., iff $R(t_1, \dots, t_n) \in \Gamma^*$.

explanation

Note that we have defined $f^{\mathfrak{M}/_\approx}$ and $R^{\mathfrak{M}/_\approx}$ for elements of $\mathrm{Trm}(\mathcal{L})/_\approx$ by referring to them as $[t]_\approx$, i.e., via *representatives* $t \in [t]_\approx$. We have to make sure that these definitions do not depend on the choice of these representatives, i.e., that for some other choices t' which determine the same equivalence classes ($[t]_\approx = [t']_\approx$), the definitions yield the same result. For instance, if R is a one-place predicate symbol, the last clause of the definition says that $[t]_\approx \in R^{\mathfrak{M}/_\approx}$ iff $\mathfrak{M} \vDash R(t)$. If for some other term t' with $t \approx t'$, $\mathfrak{M} \nvDash R(t)$, then the definition would require $[t']_\approx \notin R^{\mathfrak{M}/_\approx}$. If $t \approx t'$, then $[t]_\approx = [t']_\approx$, but we can't have both $[t]_\approx \in R^{\mathfrak{M}/_\approx}$ and $[t]_\approx \notin R^{\mathfrak{M}/_\approx}$. However, Proposition com.2 guarantees that this cannot happen.

Proposition com.5. *$\mathfrak{M}/_\approx$ is well defined, i.e., if t_1, …, t_n, t'_1, …, t'_n are closed terms, and $t_i \approx t'_i$ then*

1. *$[f(t_1, \dots, t_n)]_\approx = [f(t'_1, \dots, t'_n)]_\approx$, i.e.,*

$$f(t_1, \dots, t_n) \approx f(t'_1, \dots, t'_n)$$

and

2. *$\mathfrak{M} \vDash R(t_1, \dots, t_n)$ iff $\mathfrak{M} \vDash R(t'_1, \dots, t'_n)$, i.e.,*

$$R(t_1, \dots, t_n) \in \Gamma^* \text{ iff } R(t'_1, \dots, t'_n) \in \Gamma^*.$$

Proof. Follows from Proposition com.2 by induction on n. □

As in the case of the term model, before proving the truth lemma we need the following lemma.

fol:com:ide: lem:val-in-termmodel-factored

Lemma com.6. *Let* $\mathfrak{M} = \mathfrak{M}(\Gamma^*)$*, then* $\mathrm{Val}^{\mathfrak{M}/\approx}(t) = [t]_\approx$.

Proof. The proof is similar to that of **??**. □

Problem com.2. Complete the proof of Lemma com.6.

fol:com:ide: lem:truth

Lemma com.7. $\mathfrak{M}/_\approx \vDash \varphi$ *iff* $\varphi \in \Gamma^*$ *for all sentences* φ.

Proof. By induction on φ, just as in the proof of **??**. The only case that needs additional attention is when $\varphi \equiv t = t'$.

$$\begin{aligned}\mathfrak{M}/_\approx \vDash t = t' &\text{ iff } [t]_\approx = [t']_\approx \text{ (by definition of } \mathfrak{M}/_\approx) \\ &\text{ iff } t \approx t' \text{ (by definition of } [t]_\approx) \\ &\text{ iff } t = t' \in \Gamma^* \text{ (by definition of } \approx).\end{aligned}$$ □

Note that while $\mathfrak{M}(\Gamma^*)$ is always enumerable and infinite, $\mathfrak{M}/_\approx$ may be finite, since it may turn out that there are only finitely many classes $[t]_\approx$. This is to be expected, since Γ may contain sentences which require any structure in which they are true to be finite. For instance, $\forall x\,\forall y\, x = y$ is a consistent sentence, but is satisfied only in structures with a domain that contains exactly one element. digression

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Bibliography