## com.1 Henkin Expansion

fol:com:hen:sec

Part of the challenge in proving the completeness theorem is that the model we construct from a complete consistent set $\Gamma$ must make all the quantified formulas in $\Gamma$ true. In order to guarantee this, we use a trick due to Leon Henkin. In essence, the trick consists in expanding the language by infinitely many constant symbols and adding, for each formula with one free variable $\varphi(x)$ a formula of the form $\exists x\, \varphi(x) \to \varphi(c)$, where $c$ is one of the new constant symbols. When we construct the structure satisfying $\Gamma$, this will guarantee that each true existential sentence has a witness among the new constants. explanation

*fol:com:hen:prop:lang-exp*

**Proposition com.1.** *If $\Gamma$ is consistent in $\mathcal{L}$ and $\mathcal{L}'$ is obtained from $\mathcal{L}$ by adding a denumerable set of new constant symbols $d_0$, $d_1$, …, then $\Gamma$ is consistent in $\mathcal{L}'$.*

**Definition com.2 (Saturated set).** A set $\Gamma$ of formulas of a language $\mathcal{L}$ is *saturated* iff for each formula $\varphi(x) \in \mathrm{Frm}(\mathcal{L})$ with one free variable $x$ there is a constant symbol $c \in \mathcal{L}$ such that $\exists x\, \varphi(x) \to \varphi(c) \in \Gamma$.

The following definition will be used in the proof of the next theorem.

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**Definition com.3.** Let $\mathcal{L}'$ be as in Proposition com.1. Fix an enumeration $\varphi_0(x_0), \varphi_1(x_1), \ldots$ of all formulas $\varphi_i(x_i)$ of $\mathcal{L}'$ in which one variable ($x_i$) occurs free. We define the sentences $\theta_n$ by induction on $n$.

Let $c_0$ be the first constant symbol among the $d_i$ we added to $\mathcal{L}$ which does not occur in $\varphi_0(x_0)$. Assuming that $\theta_0$, …, $\theta_{n-1}$ have already been defined, let $c_n$ be the first among the new constant symbols $d_i$ that occurs neither in $\theta_0$, …, $\theta_{n-1}$ nor in $\varphi_n(x_n)$.

Now let $\theta_n$ be the formula $\exists x_n\, \varphi_n(x_n) \to \varphi_n(c_n)$.

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**Lemma com.4.** *Every consistent set $\Gamma$ can be extended to a saturated consistent set $\Gamma'$.*

*Proof.* Given a consistent set of sentences $\Gamma$ in a language $\mathcal{L}$, expand the language by adding a denumerable set of new constant symbols to form $\mathcal{L}'$. By Proposition com.1, $\Gamma$ is still consistent in the richer language. Further, let $\theta_i$ be as in Definition com.3. Let

$$\Gamma_0 = \Gamma$$
$$\Gamma_{n+1} = \Gamma_n \cup \{\theta_n\}$$

i.e., $\Gamma_{n+1} = \Gamma \cup \{\theta_0, \ldots, \theta_n\}$, and let $\Gamma' = \bigcup_n \Gamma_n$. $\Gamma'$ is clearly saturated.

If $\Gamma'$ were inconsistent, then for some $n$, $\Gamma_n$ would be inconsistent (Exercise: explain why). So to show that $\Gamma'$ is consistent it suffices to show, by induction on $n$, that each set $\Gamma_n$ is consistent.

The induction basis is simply the claim that $\Gamma_0 = \Gamma$ is consistent, which is the hypothesis of the theorem. For the induction step, suppose that $\Gamma_n$ is consistent but $\Gamma_{n+1} = \Gamma_n \cup \{\theta_n\}$ is inconsistent. Recall that $\theta_n$ is $\exists x_n\, \varphi_n(x_n) \to \varphi_n(c_n)$, where $\varphi_n(x_n)$ is a formula of $\mathcal{L}'$ with only the variable $x_n$ free. By the way we've chosen the $c_n$ (see Definition com.3), $c_n$ does not occur in $\varphi_n(x_n)$ nor in $\Gamma_n$.

If $\Gamma_n \cup \{\theta_n\}$ is inconsistent, then $\Gamma_n \vdash \neg\theta_n$, and hence both of the following hold:

$$\Gamma_n \vdash \exists x_n\, \varphi_n(x_n) \qquad \Gamma_n \vdash \neg\varphi_n(c_n)$$

Since $c_n$ does not occur in $\Gamma_n$ or in $\varphi_n(x_n)$, **??????????????** applies. From $\Gamma_n \vdash \neg\varphi_n(c_n)$, we obtain $\Gamma_n \vdash \forall x_n\, \neg\varphi_n(x_n)$. Thus we have that both $\Gamma_n \vdash \exists x_n\, \varphi_n(x_n)$ and $\Gamma_n \vdash \forall x_n\, \neg\varphi_n(x_n)$, so $\Gamma_n$ itself is inconsistent. (Note that $\forall x_n\, \neg\varphi_n(x_n) \vdash \neg\exists x_n\, \varphi_n(x_n)$.) Contradiction: $\Gamma_n$ was supposed to be consistent. Hence $\Gamma_n \cup \{\theta_n\}$ is consistent. □

explanation

We'll now show that *complete*, consistent sets which are saturated have the property that it contains a universally quantified sentence iff it contains all its instances and it contains an existentially quantified sentence iff it contains at least one instance. We'll use this to show that the structure we'll generate from a complete, consistent, saturated set makes all its quantified sentences true.

**Proposition com.5.** *Suppose $\Gamma$ is complete, consistent, and saturated.* *fol:com:hen: prop:saturated-instances*

1. $\exists x\, \varphi(x) \in \Gamma$ *iff* $\varphi(t) \in \Gamma$ *for at least one closed term* $t$.

2. $\forall x\, \varphi(x) \in \Gamma$ *iff* $\varphi(t) \in \Gamma$ *for all closed terms* $t$.

*Proof.*

1. First suppose that $\exists x\, \varphi(x) \in \Gamma$. Because $\Gamma$ is saturated, $(\exists x\, \varphi(x) \to \varphi(c)) \in \Gamma$ for some constant symbol $c$. By **??????????????**, item (1), and **????**, $\varphi(c) \in \Gamma$.

   For the other direction, saturation is not necessary: Suppose $\varphi(t) \in \Gamma$. Then $\Gamma \vdash \exists x\, \varphi(x)$ by **??????????????**, item (1). By **????**, $\exists x\, \varphi(x) \in \Gamma$.

2. Suppose that $\varphi(t) \in \Gamma$ for all closed terms $t$. By way of contradiction, assume $\forall x\, \varphi(x) \notin \Gamma$. Since $\Gamma$ is complete, $\neg\forall x\, \varphi(x) \in \Gamma$. By saturation, $(\exists x\, \neg\varphi(x) \to \neg\varphi(c)) \in \Gamma$ for some constant symbol $c$. By assumption, since $c$ is a closed term, $\varphi(c) \in \Gamma$. But this would make $\Gamma$ inconsistent. (Exercise: give the derivation that shows

   $$\neg\forall x\, \varphi(x), \exists x\, \neg\varphi(x) \to \neg\varphi(c), \varphi(c)$$

   is inconsistent.)

   For the reverse direction, we do not need saturation: Suppose $\forall x\, \varphi(x) \in \Gamma$. Then $\Gamma \vdash \varphi(t)$ by **??????????????**, item (2). We get $\varphi(t) \in \Gamma$ by **??**. □

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## Bibliography