

com.1 Construction of a Model

fol:com:mod:sec Right now we are not concerned about $=$, i.e., we only want to show that a explanation consistent set Γ of **sentences** not containing $=$ is satisfiable. We first extend Γ to a consistent, **complete**, and saturated set Γ^* . In this case, the definition of a model $\mathfrak{M}(\Gamma^*)$ is simple: We take the set of closed terms of \mathcal{L}' as the domain. We assign every **constant symbol** to itself, and make sure that more generally, for every closed term t , $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$. The **predicate symbols** are assigned extensions in such a way that an atomic **sentence** is true in $\mathfrak{M}(\Gamma^*)$ iff it is in Γ^* . This will obviously make all the atomic **sentences** in Γ^* true in $\mathfrak{M}(\Gamma^*)$. The rest are true provided the Γ^* we start with is consistent, complete, and saturated.

fol:com:mod: defn:termmodel **Definition com.1 (Term model).** Let Γ^* be a **complete** and consistent, saturated set of **sentences** in a language \mathcal{L} . The *term model* $\mathfrak{M}(\Gamma^*)$ of Γ^* is the **structure** defined as follows:

1. The **domain** $|\mathfrak{M}(\Gamma^*)|$ is the set of all closed terms of \mathcal{L} .
2. The interpretation of a **constant symbol** c is c itself: $c^{\mathfrak{M}(\Gamma^*)} = c$.
3. The **function symbol** f is assigned the function which, given as arguments the closed terms t_1, \dots, t_n , has as value the closed term $f(t_1, \dots, t_n)$:

$$f^{\mathfrak{M}(\Gamma^*)}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$

4. If R is an n -place **predicate symbol**, then

$$\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)} \text{ iff } R(t_1, \dots, t_n) \in \Gamma^*.$$

We will now check that we indeed have $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.

fol:com:mod: lem:val-in-termmodel **Lemma com.2.** Let $\mathfrak{M}(\Gamma^*)$ be the term model of **Definition com.1**, then $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.

Proof. The proof is by induction on t , where the base case, when t is a **constant symbol**, follows directly from the definition of the term model. For the induction step assume t_1, \dots, t_n are closed terms such that $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t_i) = t_i$ and that f is an n -ary **function symbol**. Then

$$\begin{aligned} \text{Val}^{\mathfrak{M}(\Gamma^*)}(f(t_1, \dots, t_n)) &= f^{\mathfrak{M}(\Gamma^*)}(\text{Val}^{\mathfrak{M}(\Gamma^*)}(t_1), \dots, \text{Val}^{\mathfrak{M}(\Gamma^*)}(t_n)) \\ &= f^{\mathfrak{M}(\Gamma^*)}(t_1, \dots, t_n) \\ &= f(t_1, \dots, t_n), \end{aligned}$$

and so by induction this holds for every closed term t . □

explanation A structure \mathfrak{M} may make an existentially quantified sentence $\exists x \varphi(x)$ true without there being an instance $\varphi(t)$ that it makes true. A structure \mathfrak{M} may make all instances $\varphi(t)$ of a universally quantified sentence $\forall x \varphi(x)$ true, without making $\forall x \varphi(x)$ true. This is because in general not every element of $|\mathfrak{M}|$ is the value of a closed term (\mathfrak{M} may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model $\mathfrak{M}(\Gamma^*)$ this wouldn't be necessary—because it is covered. This is the content of the next result.

Proposition com.3. *Let $\mathfrak{M}(\Gamma^*)$ be the term model of [Definition com.1](#).*

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prop:quant-termmodel

1. $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$ iff $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ for at least one closed term t .
2. $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$ iff $\mathfrak{M}(\Gamma^*) \models \varphi(t)$ for all closed terms t .

Proof. 1. By ??, $\mathfrak{M}(\Gamma^*) \models \exists x \varphi(x)$ iff for at least one variable assignment s , $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$. As $|\mathfrak{M}(\Gamma^*)|$ consists of the closed terms of \mathcal{L} , this is the case iff there is at least one closed term t such that $s(x) = t$ and $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$. By ??, $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ iff $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$, where $s(x) = t$. By ??, $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ iff $\mathfrak{M}(\Gamma^*) \models \varphi(t)$, since $\varphi(t)$ is a sentence.

2. By ??, $\mathfrak{M}(\Gamma^*) \models \forall x \varphi(x)$ iff for every variable assignment s , $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$. Recall that $|\mathfrak{M}(\Gamma^*)|$ consists of the closed terms of \mathcal{L} , so for every closed term t , $s(x) = t$ is such a variable assignment, and for any variable assignment, $s(x)$ is some closed term t . By ??, $\mathfrak{M}(\Gamma^*), s \models \varphi(x)$ iff $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$, where $s(x) = t$. By ??, $\mathfrak{M}(\Gamma^*), s \models \varphi(t)$ iff $\mathfrak{M}(\Gamma^*) \models \varphi(t)$, since $\varphi(t)$ is a sentence. \square

Lemma com.4 (Truth Lemma). *Suppose φ does not contain $=$. Then $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$.*

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lem:truth

Proof. We prove both directions simultaneously, and by induction on φ .

1. $\varphi \equiv \perp$: $\mathfrak{M}(\Gamma^*) \not\models \perp$ by definition of satisfaction. On the other hand, $\perp \notin \Gamma^*$ since Γ^* is consistent.
2. $\varphi \equiv \top$: $\mathfrak{M}(\Gamma^*) \models \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since Γ^* is consistent and complete, and $\Gamma^* \vdash \top$.
3. $\varphi \equiv R(t_1, \dots, t_n)$: $\mathfrak{M}(\Gamma^*) \models R(t_1, \dots, t_n)$ iff $\langle t_1, \dots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)}$ (by the definition of satisfaction) iff $R(t_1, \dots, t_n) \in \Gamma^*$ (by the construction of $\mathfrak{M}(\Gamma^*)$).
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \not\models \psi$ (by definition of satisfaction). By induction hypothesis, $\mathfrak{M}(\Gamma^*) \not\models \psi$ iff $\psi \notin \Gamma^*$. Since Γ^* is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg\psi \in \Gamma^*$.

5. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff we have both $\mathfrak{M}(\Gamma^*) \models \psi$ and $\mathfrak{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By ????, this is the case iff $(\psi \wedge \chi) \in \Gamma^*$.
6. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \models \psi$ or $\mathfrak{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \vee \chi) \in \Gamma^*$ (by ????).
7. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \not\models \psi$ or $\mathfrak{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by ????).
8. $\varphi \equiv \forall x \psi(x)$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \models \psi(t)$ for all terms t (**Proposition com.3**). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for all terms t , by ??, this in turn is the case iff $\forall x \varphi(x) \in \Gamma^*$.
9. $\varphi \equiv \exists x \psi(x)$: $\mathfrak{M}(\Gamma^*) \models \varphi$ iff $\mathfrak{M}(\Gamma^*) \models \psi(t)$ for at least one term t (**Proposition com.3**). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for at least one term t . By ??, this in turn is the case iff $\exists x \psi(x) \in \Gamma^*$.
□

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Bibliography