**com.1 Construction of a Model**

Right now we are not concerned about $=$, i.e., we only want to show that a consistent set $\Gamma$ of sentences not containing $=$ is satisfiable. We first extend $\Gamma$ to a consistent, complete, and saturated set $\Gamma^*$. In this case, the definition of a model $\mathfrak{M}(\Gamma^*)$ is simple: We take the set of closed terms of $\mathcal{L}'$ as the domain. We assign every constant symbol to itself, and make sure that more generally, for every closed term $t$, $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$. The predicate symbols are assigned extensions in such a way that an atomic sentence is true in $\mathfrak{M}(\Gamma^*)$ iff it is in $\Gamma^*$. This will obviously make all the atomic sentences in $\Gamma^*$ true in $\mathfrak{M}(\Gamma^*)$. The rest are true provided the $\Gamma^*$ we start with is consistent, complete, and saturated.

**Definition com.1 (Term model).** Let $\Gamma^*$ be a complete and consistent, saturated set of sentences in a language $\mathcal{L}$. The term model $\mathfrak{M}(\Gamma^*)$ is the structure defined as follows:

1. The domain $|\mathfrak{M}(\Gamma^*)|$ is the set of all closed terms of $\mathcal{L}$.
2. The interpretation of a constant symbol $c$ is $c$: $c^{\mathfrak{M}(\Gamma^*)} = c$.
3. The function symbol $f$ is assigned the function which, given as arguments the closed terms $t_1, \ldots, t_n$, has as value the closed term $f(t_1, \ldots, t_n)$:
   $$f^{\mathfrak{M}(\Gamma^*)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$
4. If $R$ is an $n$-place predicate symbol, then
   $$\langle t_1, \ldots, t_n \rangle \in R^{\mathfrak{M}(\Gamma^*)} \text{ iff } R(t_1, \ldots, t_n) \in \Gamma^*.$$  

We will now check that we indeed have $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.

**Lemma com.2.** Let $\mathfrak{M}(\Gamma^*)$ be the term model of Definition com.1, then $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t) = t$.

**Proof.** The proof is by induction on $t$, where the base case, when $t$ is a constant symbol, follows directly from the definition of the term model. For the induction step assume $t_1, \ldots, t_n$ are closed terms such that $\text{Val}^{\mathfrak{M}(\Gamma^*)}(t_i) = t_i$ and that $f$ is an $n$-ary function symbol. Then

$$\text{Val}^{\mathfrak{M}(\Gamma^*)}(f(t_1, \ldots, t_n)) = f^{\mathfrak{M}(\Gamma^*)}(\text{Val}^{\mathfrak{M}(\Gamma^*)}(t_1), \ldots, \text{Val}^{\mathfrak{M}(\Gamma^*)}(t_n))$$

$$= f^{\mathfrak{M}(\Gamma^*)}(t_1, \ldots, t_n)$$

$$= f(t_1, \ldots, t_n),$$

and so by induction this holds for every closed term $t$. 

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A structure $\mathcal{M}$ may make an existentially quantified sentence $\exists x \varphi(x)$ true without there being an instance $\varphi(t)$ that it makes true. A structure $\mathcal{M}$ may make all instances $\varphi(t)$ of a universally quantified sentence $\forall x \varphi(x)$ true, without making $\forall x \varphi(x)$ true. This is because in general not every element of $|\mathcal{M}|$ is the value of a closed term ($\mathcal{M}$ may not be covered). This is the reason the satisfaction relation is defined via variable assignments. However, for our term model $\mathcal{M}(\Gamma^*)$ this wouldn’t be necessary—because it is covered. This is the content of the next result.

**Proposition com.3.** Let $\mathcal{M}(\Gamma^*)$ be the term model of Definition com.1.

1. $\mathcal{M}(\Gamma^*) \models \exists x \varphi(x)$ iff $\mathcal{M}(\Gamma^*) \models \varphi(t)$ for at least one closed term $t$.

2. $\mathcal{M}(\Gamma^*) \models \forall x \varphi(x)$ iff $\mathcal{M}(\Gamma^*) \models \varphi(t)$ for all closed terms $t$.

**Proof.**

1. By $??$, $\mathcal{M}(\Gamma^*) \models \exists x \varphi(x)$ iff for at least one variable assignment $s$, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$. As $|\mathcal{M}(\Gamma^*)|$ consists of the closed terms of $\mathcal{L}$, this is the case iff there is at least one closed term $t$ such that $s(x) = t$ and $\mathcal{M}(\Gamma^*), s \models \varphi(x)$. By $??$, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$ iff $\mathcal{M}(\Gamma^*), s \models \varphi(t)$, where $s(x) = t$. By $??$, $\mathcal{M}(\Gamma^*), s \models \varphi(t)$ iff $\mathcal{M}(\Gamma^*) \models \varphi(t)$, since $\varphi(t)$ is a sentence.

2. By $??$, $\mathcal{M}(\Gamma^*) \models \forall x \varphi(x)$ iff for every variable assignment $s$, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$. Recall that $|\mathcal{M}(\Gamma^*)|$ consists of the closed terms of $\mathcal{L}$, so for every closed term $t$, $s(x) = t$ is such a variable assignment, and for any variable assignment, $s(x)$ is some closed term $t$. By $??$, $\mathcal{M}(\Gamma^*), s \models \varphi(x)$ iff $\mathcal{M}(\Gamma^*), s \models \varphi(t)$, where $s(x) = t$. By $??$, $\mathcal{M}(\Gamma^*), s \models \varphi(t)$ iff $\mathcal{M}(\Gamma^*) \models \varphi(t)$, since $\varphi(t)$ is a sentence.

**Lemma com.4 (Truth Lemma).** Suppose $\varphi$ does not contain $\Rightarrow$. Then $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\varphi \in \Gamma^*$.

**Proof.** We prove both directions simultaneously, and by induction on $\varphi$.

1. $\varphi \equiv \bot$: $\mathcal{M}(\Gamma^*) \not\models \bot$ by definition of satisfaction. On the other hand, $\bot \notin \Gamma^*$ since $\Gamma^*$ is consistent.

2. $\varphi \equiv \top$: $\mathcal{M}(\Gamma^*) \models \top$ by definition of satisfaction. On the other hand, $\top \in \Gamma^*$ since $\Gamma^*$ is consistent and complete, and $\Gamma^* \models \top$.

3. $\varphi \equiv R(t_1, \ldots, t_n)$: $\mathcal{M}(\Gamma^*) \models R(t_1, \ldots, t_n)$ iff $(t_1, \ldots, t_n) \in R^{\mathcal{M}(\Gamma^*)}$ (by the definition of satisfaction) iff $R(t_1, \ldots, t_n) \in \Gamma^*$ (by the construction of $\mathcal{M}(\Gamma^*)$).

4. $\varphi \equiv \neg \psi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \not\models \psi$ (by definition of satisfaction). By induction hypothesis, $\mathcal{M}(\Gamma^*) \not\models \psi$ iff $\psi \notin \Gamma^*$. Since $\Gamma^*$ is consistent and complete, $\psi \notin \Gamma^*$ iff $\neg \psi \in \Gamma^*$.
5. $\varphi \equiv \psi \land \chi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff we have both $\mathcal{M}(\Gamma^*) \models \psi$ and $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff both $\psi \in \Gamma^*$ and $\chi \in \Gamma^*$ (by the induction hypothesis). By $\square$, this is the case iff $(\psi \land \chi) \in \Gamma^*$.

6. $\varphi \equiv \psi \lor \chi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \models \psi$ or $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \lor \chi) \in \Gamma^*$ (by $\square$).

7. $\varphi \equiv \psi \rightarrow \chi$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \not\models \psi$ or $\mathcal{M}(\Gamma^*) \models \chi$ (by definition of satisfaction) iff $\psi \notin \Gamma^*$ or $\chi \in \Gamma^*$ (by induction hypothesis). This is the case iff $(\psi \rightarrow \chi) \in \Gamma^*$ (by $\square$).

8. $\varphi \equiv \forall x \, \psi(x)$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \models \psi(t)$ for all terms $t$ (Proposition com.3). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for all terms $t$, by $\square$, this in turn is the case iff $\forall x \, \varphi(x) \in \Gamma^*$.

9. $\varphi \equiv \exists x \, \psi(x)$: $\mathcal{M}(\Gamma^*) \models \varphi$ iff $\mathcal{M}(\Gamma^*) \models \psi(t)$ for at least one term $t$ (Proposition com.3). By induction hypothesis, this is the case iff $\psi(t) \in \Gamma^*$ for at least one term $t$. By $\square$, this in turn is the case iff $\exists x \, \psi(x) \in \Gamma^*$.

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Bibliography