

## axd.1 Examples of Derivations

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sec

**Example axd.1.** Suppose we want to prove  $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$ . Clearly, this is not an instance of any of our axioms, so we have to use the MP rule to **derive** it. Our only rule is MP, which given  $\varphi$  and  $\varphi \rightarrow \psi$  allows us to justify  $\psi$ . One strategy would be to use ?? with  $\varphi$  being  $\neg\theta$ ,  $\psi$  being  $\alpha$ , and  $\chi$  being  $\theta \rightarrow \alpha$ , i.e., the instance

$$(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha))).$$

Why? Two applications of MP yield the last part, which is what we want. And we easily see that  $\neg\theta \rightarrow (\theta \rightarrow \alpha)$  is an instance of ??, and  $\alpha \rightarrow (\theta \rightarrow \alpha)$  is an instance of ??. So our derivation is:

1.  $\neg\theta \rightarrow (\theta \rightarrow \alpha)$  ??
2.  $(\neg\theta \rightarrow (\theta \rightarrow \alpha)) \rightarrow$   
     $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$  ??
3.  $((\alpha \rightarrow (\theta \rightarrow \alpha)) \rightarrow ((\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)))$  1, 2, MP
4.  $\alpha \rightarrow (\theta \rightarrow \alpha)$  ??
5.  $(\neg\theta \vee \alpha) \rightarrow (\theta \rightarrow \alpha)$  3, 4, MP

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**Example axd.2.** Let's try to find a **derivation** of  $\theta \rightarrow \theta$ . It is not an instance of an axiom, so we have to use MP to **derive** it. ?? is an axiom of the form  $\varphi \rightarrow \psi$  to which we could apply MP. To be useful, of course, the  $\psi$  which MP would justify as a correct step in this case would have to be  $\theta \rightarrow \theta$ , since this is what we want to **derive**. That means  $\varphi$  would also have to be  $\theta$ , i.e., we might look at this instance of ??:

$$\theta \rightarrow (\theta \rightarrow \theta)$$

In order to apply MP, we would also need to justify the corresponding second premise, namely  $\varphi$ . But in our case, that would be  $\theta$ , and we won't be able to **derive**  $\theta$  by itself. So we need a different strategy.

The other axiom involving just  $\rightarrow$  is ??, i.e.,

$$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$$

We could get to the last nested conditional by applying MP twice. Again, that would mean that we want an instance of ?? where  $\varphi \rightarrow \chi$  is  $\theta \rightarrow \theta$ , the **formula** we are aiming for. Then of course,  $\varphi$  and  $\chi$  are both  $\theta$ . How should we pick  $\psi$  so that both  $\varphi \rightarrow (\psi \rightarrow \chi)$  and  $\varphi \rightarrow \psi$ , i.e., in our case  $\theta \rightarrow (\psi \rightarrow \theta)$  and  $\theta \rightarrow \psi$ , are also **derivable**? Well, the first of these is already an instance of ??, whatever we decide  $\psi$  to be. And  $\theta \rightarrow \psi$  would be another instance of ?? if  $\psi$  were  $(\theta \rightarrow \theta)$ . So, our derivation is:

1.  $\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)$  ??
2.  $(\theta \rightarrow ((\theta \rightarrow \theta) \rightarrow \theta)) \rightarrow$   
     $((\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta))$  ??
3.  $(\theta \rightarrow (\theta \rightarrow \theta)) \rightarrow (\theta \rightarrow \theta)$  1, 2, MP
4.  $\theta \rightarrow (\theta \rightarrow \theta)$  ??
5.  $\theta \rightarrow \theta$  3, 4, MP

**Example axd.3.** Sometimes we want to show that there is a derivation of some formula from some other formulas  $\Gamma$ . For instance, let's show that we can derive  $\varphi \rightarrow \chi$  from  $\Gamma = \{\varphi \rightarrow \psi, \psi \rightarrow \chi\}$ . fol:axd:prop:ex:chain

- |    |  |          |
|----|--|----------|
| 1. | $\varphi \rightarrow \psi$   | HYP      |
| 2. | $\psi \rightarrow \chi$  | HYP      |
| 3. | $(\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$  | ??       |
| 4. | $\varphi \rightarrow (\psi \rightarrow \chi)$  | 2, 3, MP |
| 5. | $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow$<br>$((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ | ??       |
| 6. | $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$  | 4, 5, MP |
| 7. | $\varphi \rightarrow \chi$   | 1, 6, MP |

The lines labelled “HYP” (for “hypothesis”) indicate that the formula on that line is an element of  $\Gamma$ .

**Proposition axd.4.** *If  $\Gamma \vdash \varphi \rightarrow \psi$  and  $\Gamma \vdash \psi \rightarrow \chi$ , then  $\Gamma \vdash \varphi \rightarrow \chi$*  fol:axd:prop:prop:chain

*Proof.* Suppose  $\Gamma \vdash \varphi \rightarrow \psi$  and  $\Gamma \vdash \psi \rightarrow \chi$ . Then there is a derivation of  $\varphi \rightarrow \psi$  from  $\Gamma$ ; and a derivation of  $\psi \rightarrow \chi$  from  $\Gamma$  as well. Combine these into a single derivation by concatenating them. Now add lines 3–7 of the derivation in the preceding example. This is a derivation of  $\varphi \rightarrow \chi$ —which is the last line of the new derivation—from  $\Gamma$ . Note that the justifications of lines 4 and 7 remain valid if the reference to line number 2 is replaced by reference to the last line of the derivation of  $\varphi \rightarrow \psi$ , and reference to line number 1 by reference to the last line of the derivation of  $\psi \rightarrow \chi$ . □

**Problem axd.1.** Show that the following hold by exhibiting derivations from the axioms:

1.  $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
2.  $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
3.  $\neg(\varphi \vee \psi) \rightarrow \neg\varphi$

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## Bibliography