

## axd.1 Derivability and Consistency

fol:axd:prv:sec We will now establish a number of properties of the **derivability** relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

fol:axd:prv:prop:provability-contr **Proposition axd.1.** *If  $\Gamma \vdash \varphi$  and  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* If  $\Gamma \cup \{\varphi\}$  is inconsistent, then  $\Gamma \cup \{\varphi\} \vdash \perp$ . By ??,  $\Gamma \vdash \psi$  for every  $\psi \in \Gamma$ . Since also  $\Gamma \vdash \varphi$  by hypothesis,  $\Gamma \vdash \psi$  for every  $\psi \in \Gamma \cup \{\varphi\}$ . By ??,  $\Gamma \vdash \perp$ , i.e.,  $\Gamma$  is inconsistent.  $\square$

fol:axd:prv:prop:prov-incons **Proposition axd.2.**  *$\Gamma \vdash \varphi$  iff  $\Gamma \cup \{\neg\varphi\}$  is inconsistent.*

*Proof.* First suppose  $\Gamma \vdash \varphi$ . Then  $\Gamma \cup \{\neg\varphi\} \vdash \varphi$  by ??.  $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$  by ??. We also have  $\vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$  by ??. So by two applications of ??, we have  $\Gamma \cup \{\neg\varphi\} \vdash \perp$ .

Now assume  $\Gamma \cup \{\neg\varphi\}$  is inconsistent, i.e.,  $\Gamma \cup \{\neg\varphi\} \vdash \perp$ . By the deduction theorem,  $\Gamma \vdash \neg\varphi \rightarrow \perp$ .  $\Gamma \vdash (\neg\varphi \rightarrow \perp) \rightarrow \neg\neg\varphi$  by ??, so  $\Gamma \vdash \neg\neg\varphi$  by ??. Since  $\Gamma \vdash \neg\neg\varphi \rightarrow \varphi$  (??), we have  $\Gamma \vdash \varphi$  by ?? again.  $\square$

**Problem axd.1.** Prove that  $\Gamma \vdash \neg\varphi$  iff  $\Gamma \cup \{\varphi\}$  is inconsistent.

fol:axd:prv:prop:explicit-inc **Proposition axd.3.** *If  $\Gamma \vdash \varphi$  and  $\neg\varphi \in \Gamma$ , then  $\Gamma$  is inconsistent.*

*Proof.*  $\Gamma \vdash \neg\varphi \rightarrow (\varphi \rightarrow \perp)$  by ??.  $\Gamma \vdash \perp$  by two applications of ??.  $\square$

fol:axd:prv:prop:provability-exhaustive **Proposition axd.4.** *If  $\Gamma \cup \{\varphi\}$  and  $\Gamma \cup \{\neg\varphi\}$  are both inconsistent, then  $\Gamma$  is inconsistent.*

*Proof.* Exercise.  $\square$

**Problem axd.2.** Prove **Proposition axd.4**

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## Bibliography