axd.1 Derivability and Consistency

We will now establish a number of properties of the derivability relation. They are independently interesting, but each will play a role in the proof of the completeness theorem.

Proposition axd.1. If $\Gamma \vdash \varphi$ and $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma$ is inconsistent.

Proof. If $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$. By ??, $\Gamma \vdash \psi$ for every $\psi \in \Gamma$. Since also $\Gamma \vdash \varphi$ by hypothesis, $\Gamma \vdash \psi$ for every $\psi \in \Gamma \cup \{\varphi\}$. By ??, $\Gamma \vdash \bot$, i.e., $\Gamma$ is inconsistent. $\square$

Proposition axd.2. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg \varphi\}$ is inconsistent.

Proof. First suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\neg \varphi\} \vdash \varphi$ by ??, $\Gamma \cup \{\neg \varphi\} \vdash \neg \varphi$ by ??.. We also have $\vdash \neg \varphi \rightarrow (\varphi \rightarrow \bot)$ by ??.. So by two applications of ??, we have $\Gamma \cup \{\neg \varphi\} \vdash \bot$.

Now assume $\Gamma \cup \{\neg \varphi\}$ is inconsistent, i.e., $\Gamma \cup \{\neg \varphi\} \vdash \bot$. By the deduction theorem, $\Gamma \vdash \neg \varphi \rightarrow \bot$. $\Gamma \vdash (\neg \varphi \rightarrow \bot) \rightarrow \neg \neg \varphi$ by ??, so $\Gamma \vdash \neg \neg \varphi$ by ??.. Since $\Gamma \vdash \neg \neg \varphi \rightarrow \varphi$ (??), we have $\Gamma \vdash \varphi$ by ?? again. $\square$

Problem axd.1. Prove that $\Gamma \vdash \neg \varphi$ iff $\Gamma \cup \{\varphi\}$ is inconsistent.

Proposition axd.3. If $\Gamma \vdash \varphi$ and $\neg \varphi \in \Gamma$, then $\Gamma$ is inconsistent.

Proof. $\Gamma \vdash \neg \varphi \rightarrow (\varphi \rightarrow \bot)$ by ??.. $\Gamma \vdash \bot$ by two applications of ??.. $\square$

Proposition axd.4. If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg \varphi\}$ are both inconsistent, then $\Gamma$ is inconsistent.

Proof. Exercise. $\square$

Problem axd.2. Prove Proposition axd.4

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Bibliography