Just as we’ve defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding proof-theoretic notions. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the soundness and completeness theorems.

Definition axd.1 (Derivability). A formula \( \varphi \) is derivable from \( \Gamma \), written \( \Gamma \vdash \varphi \), if there is a derivation from \( \Gamma \) ending in \( \varphi \).

Definition axd.2 (Theorems). A formula \( \varphi \) is a theorem if there is a derivation of \( \varphi \) from the empty set. We write \( \vdash \varphi \) if \( \varphi \) is a theorem and \( \nvdash \varphi \) if it is not.

Definition axd.3 (Consistency). A set \( \Gamma \) of formulas is consistent if and only if \( \Gamma \nvdash \bot \); it is inconsistent otherwise.

Proposition axd.4 (Reflexivity). If \( \varphi \in \Gamma \), then \( \Gamma \vdash \varphi \).

Proof. The formula \( \varphi \) by itself is a derivation of \( \varphi \) from \( \Gamma \).

Proposition axd.5 (Monotonicity). If \( \Gamma \subseteq \Delta \) and \( \Gamma \vdash \varphi \), then \( \Delta \vdash \varphi \).

Proof. Any derivation of \( \varphi \) from \( \Gamma \) is also a derivation of \( \varphi \) from \( \Delta \).

Proposition axd.6 (Transitivity). If \( \Gamma \vdash \varphi \) and \( \{ \varphi \} \cup \Delta \vdash \psi \), then \( \Gamma \cup \Delta \vdash \psi \).

Proof. Suppose \( \{ \varphi \} \cup \Delta \vdash \psi \). Then there is a derivation \( \psi_1, \ldots, \psi_l = \psi \) from \( \{ \varphi \} \cup \Delta \). Some of the steps in that derivation will be correct because of a rule which refers to a prior line \( \psi_i = \varphi \). By hypothesis, there is a derivation of \( \varphi \) from \( \Gamma \), i.e., a derivation \( \varphi_1, \ldots, \varphi_k = \varphi \) where every \( \varphi_i \) is an axiom, an element of \( \Gamma \), or correct by a rule of inference. Now consider the sequence

\[
\varphi_1, \ldots, \varphi_k = \varphi, \psi_1, \ldots, \psi_l = \psi.
\]

This is a correct derivation of \( \psi \) from \( \Gamma \cup \Delta \) since every \( B_i = \varphi \) is now justified by the same rule which justifies \( \varphi_k = \varphi \).

Note that this means that in particular if \( \Gamma \vdash \varphi \) and \( \varphi \vdash \psi \), then \( \Gamma \vdash \psi \). It follows also that if \( \varphi_1, \ldots, \varphi_n \vdash \psi \) and \( \Gamma \vdash \varphi_i \) for each \( i \), then \( \Gamma \vdash \psi \).

Proposition axd.7. \( \Gamma \) is inconsistent iff \( \Gamma \vdash \varphi \) for every \( \varphi \).

Proof. Exercise.

Problem axd.1. Prove Proposition axd.7.
Proposition axd.8 (Compactness).

1. If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

2. If every finite subset of $\Gamma$ is consistent, then $\Gamma$ is consistent.

Proof. 1. If $\Gamma \vdash \varphi$, then there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each $\varphi_i$ is either a logical axiom, an element of $\Gamma$ or follows from previous formulas by modus ponens. Take $\Gamma_0$ to be those $\varphi_i$ which are in $\Gamma$. Then the derivation is likewise a derivation from $\Gamma_0$, and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$. \hfill \Box

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Bibliography