axd.1 Proof-Theoretic Notions

fol:axd:ptn: sec Just as we've defined a number of important semantic notions (validity, entailment, satisfiabilty), we now define corresponding *proof-theoretic notions*. These are not defined by appeal to satisfaction of sentences in structures, but by appeal to the derivability or non-derivability of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*. explanation

Definition axd.1 (Derivability). A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition axd.2 (Theorems). A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Definition axd.3 (Consistency). A set Γ of formulas is *consistent* if and only if $\Gamma \nvdash \bot$; it is *inconsistent* otherwise.

fol:axd:ptn: prop:reflexivity **Proposition axd.4 (Reflexivity).** *If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.*

Proof. The formula φ by itself is a derivation of φ from Γ. □

fol:axd:ptn: prop:monotony **Proposition axd.5 (Monotony).** *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$, then $\Delta \vdash \varphi$.*

Proof. Any derivation of φ from Γ is also a derivation of φ from Δ. □

fol:axd:ptn: prop:transitivity **Proposition axd.6 (Transitivity).** *If $\Gamma \vdash \varphi$ and $\{\varphi\} \cup \Delta \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$.*

Proof. Suppose $\{\varphi\} \cup \Delta \vdash \psi$. Then there is a derivation ψ_1, ..., $\psi_l = \psi$ from $\{\varphi\} \cup \Delta$. Some of the steps in that derivation will be correct because of a rule which refers to a prior line $\psi_i = \varphi$. By hypothesis, there is a derivation of φ from Γ, i.e., a derivation φ_1, ..., $\varphi_k = \varphi$ where every φ_i is an axiom, an element of Γ, or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \ldots, \varphi_k = \varphi, \psi_1, \ldots, \psi_l = \psi.$$

This is a correct derivation of ψ from $\Gamma \cup \Delta$ since every $B_i = \varphi$ is now justified by the same rule which justifies $\varphi_k = \varphi$. □

Note that this means that in particular if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. It follows also that if $\varphi_1, \ldots, \varphi_n \vdash \psi$ and $\Gamma \vdash \varphi_i$ for each i, then $\Gamma \vdash \psi$.

fol:axd:ptn: prop:incons **Proposition axd.7.** *Γ is inconsistent iff $\Gamma \vdash \varphi$ for every φ.*

Proof. Exercise. □

Problem axd.1. Prove Proposition axd.7.

Proposition axd.8 (Compactness). fol:axd:ptn: prop:proves-compact

1. *If $\Gamma \vdash \varphi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.*

2. *If every finite subset of Γ is consistent, then Γ is consistent.*

Proof.

1. If $\Gamma \vdash \varphi$, then there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ so that $\varphi \equiv \varphi_n$ and each φ_i is either a logical axiom, an element of Γ or follows from previous formulas by modus ponens. Take Γ_0 to be those φ_i which are in Γ. Then the derivation is likewise a derivation from Γ_0, and so $\Gamma_0 \vdash \varphi$.

2. This is the contrapositive of (1) for the special case $\varphi \equiv \bot$. □

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Bibliography