

## axd.1 Proof-Theoretic Notions

fol:axd:ptn:  
sec Just as we've defined a number of important semantic notions (validity, entailment, satisfiability), we now define corresponding *proof-theoretic notions*. explanation These are not defined by appeal to satisfaction of **sentences** in **structures**, but by appeal to the **derivability** or **non-derivability** of certain formulas. It was an important discovery that these notions coincide. That they do is the content of the *soundness* and *completeness theorems*.

**Definition axd.1 (Derivability).** A formula  $\varphi$  is *derivable* from  $\Gamma$ , written  $\Gamma \vdash \varphi$ , if there is a **derivation** from  $\Gamma$  ending in  $\varphi$ .

**Definition axd.2 (Theorems).** A formula  $\varphi$  is a *theorem* if there is a **derivation** of  $\varphi$  from the empty set. We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Definition axd.3 (Consistency).** A set  $\Gamma$  of **formulas** is *consistent* if and only if  $\Gamma \not\vdash \perp$ ; it is *inconsistent* otherwise.

fol:axd:ptn:  
prop:reflexivity **Proposition axd.4 (Reflexivity).** If  $\varphi \in \Gamma$ , then  $\Gamma \vdash \varphi$ .

*Proof.* The **formula**  $\varphi$  by itself is a **derivation** of  $\varphi$  from  $\Gamma$ . □

fol:axd:ptn:  
prop:monotonicity **Proposition axd.5 (Monotonicity).** If  $\Gamma \subseteq \Delta$  and  $\Gamma \vdash \varphi$ , then  $\Delta \vdash \varphi$ .

*Proof.* Any **derivation** of  $\varphi$  from  $\Gamma$  is also a **derivation** of  $\varphi$  from  $\Delta$ . □

fol:axd:ptn:  
prop:transitivity **Proposition axd.6 (Transitivity).** If  $\Gamma \vdash \varphi$  and  $\{\varphi\} \cup \Delta \vdash \psi$ , then  $\Gamma \cup \Delta \vdash \psi$ .

*Proof.* Suppose  $\{\varphi\} \cup \Delta \vdash \psi$ . Then there is a **derivation**  $\psi_1, \dots, \psi_l = \psi$  from  $\{\varphi\} \cup \Delta$ . Some of the steps in that derivation will be correct because of a rule which refers to a prior line  $\psi_i = \varphi$ . By hypothesis, there is a **derivation** of  $\varphi$  from  $\Gamma$ , i.e., a **derivation**  $\varphi_1, \dots, \varphi_k = \varphi$  where every  $\varphi_i$  is an axiom, an **element** of  $\Gamma$ , or correct by a rule of inference. Now consider the sequence

$$\varphi_1, \dots, \varphi_k = \varphi, \psi_1, \dots, \psi_l = \psi.$$

This is a correct **derivation** of  $\psi$  from  $\Gamma \cup \Delta$  since every  $B_i = \varphi$  is now justified by the same rule which justifies  $\varphi_k = \varphi$ . □

Note that this means that in particular if  $\Gamma \vdash \varphi$  and  $\varphi \vdash \psi$ , then  $\Gamma \vdash \psi$ . It follows also that if  $\varphi_1, \dots, \varphi_n \vdash \psi$  and  $\Gamma \vdash \varphi_i$  for each  $i$ , then  $\Gamma \vdash \psi$ .

fol:axd:ptn:  
prop:incons **Proposition axd.7.**  $\Gamma$  is *inconsistent* iff  $\Gamma \vdash \varphi$  for every  $\varphi$ .

*Proof.* Exercise. □

**Problem axd.1.** Prove **Proposition axd.7**.

**Proposition axd.8 (Compactness).**

*fol:axd:ptn:  
prop:proves-compact*

1. If  $\Gamma \vdash \varphi$  then there is a finite subset  $\Gamma_0 \subseteq \Gamma$  such that  $\Gamma_0 \vdash \varphi$ .
2. If every finite subset of  $\Gamma$  is consistent, then  $\Gamma$  is consistent.

*Proof.* 1. If  $\Gamma \vdash \varphi$ , then there is a finite sequence of formulas  $\varphi_1, \dots, \varphi_n$  so that  $\varphi \equiv \varphi_n$  and each  $\varphi_i$  is either a logical axiom, an element of  $\Gamma$  or follows from previous formulas by modus ponens. Take  $\Gamma_0$  to be those  $\varphi_i$  which are in  $\Gamma$ . Then the derivation is likewise a derivation from  $\Gamma_0$ , and so  $\Gamma_0 \vdash \varphi$ .

2. This is the contrapositive of (1) for the special case  $\varphi \equiv \perp$ . □

## Photo Credits

## Bibliography