

axd.1 The Deduction Theorem

fol:axd:ded:
sec As we've seen, giving **derivations** in an axiomatic system is cumbersome, and **derivations** may be hard to find. Rather than actually write out long lists of **formulas**, it is generally easier to argue that such **derivations** exist, by making use of a few simple results. We've already established three such results: ?? says we can always assert that $\Gamma \vdash \varphi$ when we know that $\varphi \in \Gamma$. ?? says that if $\Gamma \vdash \varphi$ then also $\Gamma \cup \{\psi\} \vdash \varphi$. And ?? implies that if $\Gamma \vdash \varphi$ and $\varphi \vdash \psi$, then $\Gamma \vdash \psi$. Here's another simple result, a "meta"-version of modus ponens:

fol:axd:ded:
prop:mp **Proposition axd.1.** *If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$, then $\Gamma \vdash \psi$.*

Proof. We have that $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$:

1. φ Hyp.
2. $\varphi \rightarrow \psi$ Hyp.
3. ψ 1, 2, MP

By ??, $\Gamma \vdash \psi$. □

The most important result we'll use in this context is the deduction theorem:

fol:axd:ded:
thm:deduction-thm **Theorem axd.2 (Deduction Theorem).** *$\Gamma \cup \{\varphi\} \vdash \psi$ if and only if $\Gamma \vdash \varphi \rightarrow \psi$.*

Proof. The "if" direction is immediate. If $\Gamma \vdash \varphi \rightarrow \psi$ then also $\Gamma \cup \{\varphi\} \vdash \varphi \rightarrow \psi$ by ??. Also, $\Gamma \cup \{\varphi\} \vdash \varphi$ by ??. So, by **Proposition axd.1**, $\Gamma \cup \{\varphi\} \vdash \psi$.

For the "only if" direction, we proceed by induction on the length of the **derivation** of ψ from $\Gamma \cup \{\varphi\}$.

For the induction basis, we prove the claim for every **derivation** of length 1. A **derivation** of ψ from $\Gamma \cup \{\varphi\}$ of length 1 consists of ψ by itself; and if it is correct ψ is either $\in \Gamma \cup \{\varphi\}$ or is an axiom. If $\psi \in \Gamma$ or is an axiom, then $\Gamma \vdash \psi$. We also have that $\Gamma \vdash \psi \rightarrow (\varphi \rightarrow \psi)$ by ??, and **Proposition axd.1** gives $\Gamma \vdash \varphi \rightarrow \psi$. If $\psi \in \{\varphi\}$ then $\Gamma \vdash \varphi \rightarrow \psi$ because then last **sentence** $\varphi \rightarrow \psi$ is the same as $\varphi \rightarrow \varphi$, and we have **derived** that in ??.

For the inductive step, suppose a **derivation** of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by modus ponens. (If it is not justified by modus ponens, $\psi \in \Gamma$, $\psi \equiv \varphi$, or ψ is an axiom, and the same reasoning as in the induction basis applies.) Then some previous steps in the **derivation** are $\chi \rightarrow \psi$ and χ , for some **formula** χ , i.e., $\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \psi$ and $\Gamma \cup \{\varphi\} \vdash \chi$, and the respective derivations are shorter, so the inductive hypothesis applies to them. We thus have both:

$$\begin{aligned}\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \psi); \\ \Gamma \vdash \varphi \rightarrow \chi.\end{aligned}$$

But also

$$\Gamma \vdash (\varphi \rightarrow (\chi \rightarrow \psi)) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow \psi)),$$

by ??, and two applications of **Proposition axd.1** give $\Gamma \vdash \varphi \rightarrow \psi$, as required. □

Notice how ?? and ?? were chosen precisely so that the Deduction Theorem would hold.

The following are some useful facts about **derivability**, which we leave as exercises.

Proposition axd.3.

1. $\vdash (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$;
2. *If $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ then $\Gamma \cup \{\psi\} \vdash \varphi$ (Contraposition);*
3. $\{\varphi, \neg\varphi\} \vdash \psi$ (*Ex Falso Quodlibet, Explosion*);
4. $\{\neg\neg\varphi\} \vdash \varphi$ (*Double Negation Elimination*);
5. *If $\Gamma \vdash \neg\neg\varphi$ then $\Gamma \vdash \varphi$;*

*fol:axd:ded:
prop:derivfacts
fol:axd:ded:
derivfacts:a
derivfacts:b
fol:axd:ded:
derivfacts:c
fol:axd:ded:
derivfacts:d
fol:axd:ded:
derivfacts:e*

Problem axd.1. Prove **Proposition axd.3**

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Bibliography