

## axd.1 The Deduction Theorem with Quantifiers

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thm:deduction-thm-q

**Theorem axd.1 (Deduction Theorem).** *If  $\Gamma \cup \{\varphi\} \vdash \psi$ , then  $\Gamma \vdash \varphi \rightarrow \psi$ .*

*Proof.* We again proceed by induction on the length of the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$ .

The proof of the induction basis is identical to that in the proof of ??.

For the inductive step, suppose again that the **derivation** of  $\psi$  from  $\Gamma \cup \{\varphi\}$  ends with a step  $\psi$  which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of ??. If the inference rule is QR, we know that  $\psi \equiv \chi \rightarrow \forall x \theta(x)$  and a **formula** of the form  $\chi \rightarrow \theta(a)$  appears earlier in the **derivation**, where  $a$  does not occur in  $\chi$ ,  $\varphi$ , or  $\Gamma$ . We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \theta(a),$$

and the induction hypothesis applies, i.e., we have that

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \theta(a)).$$

By

$$\vdash (\varphi \rightarrow (\chi \rightarrow \theta(a))) \rightarrow ((\varphi \wedge \chi) \rightarrow \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \theta(a).$$

Since the eigenvariable condition still applies, we can add a step to this **derivation** justified by QR, and get

$$\Gamma \vdash (\varphi \wedge \chi) \rightarrow \forall x \theta(x).$$

We also have

$$\vdash ((\varphi \wedge \chi) \rightarrow \forall x \theta(x)) \rightarrow (\varphi \rightarrow (\chi \rightarrow \forall x \theta(x))),$$

so by modus ponens,

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \forall x \theta(x)),$$

i.e.,  $\Gamma \vdash \psi$ .

We leave the case where  $\psi$  is justified by the rule QR, but is of the form  $\exists x \theta(x) \rightarrow \chi$ , as an exercise.  $\square$

**Problem axd.1.** Complete the proof of **Theorem axd.1**.

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**Bibliography**