axd.1 The Deduction Theorem with Quantifiers

fol:axd:ddq:sec

fol:axd:ddq:thm:deduction-thm-q

Theorem axd.1 (Deduction Theorem). *If $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \rightarrow \psi$.*

Proof. We again proceed by induction on the length of the derivation of ψ from $\Gamma \cup \{\varphi\}$.

The proof of the induction basis is identical to that in the proof of **??**.

For the inductive step, suppose again that the derivation of ψ from $\Gamma \cup \{\varphi\}$ ends with a step ψ which is justified by an inference rule. If the inference rule is modus ponens, we proceed as in the proof of **??**. If the inference rule is QR, we know that $\psi \equiv \chi \rightarrow \forall x\, \theta(x)$ and a formula of the form $\chi \rightarrow \theta(a)$ appears earlier in the derivation, where a does not occur in χ, φ, or Γ. We thus have that

$$\Gamma \cup \{\varphi\} \vdash \chi \rightarrow \theta(a),$$

and the induction hypothesis applies, i.e., we have that

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \theta(a)).$$

By

$$\vdash (\varphi \rightarrow (\chi \rightarrow \theta(a))) \rightarrow ((\varphi \land \chi) \rightarrow \theta(a))$$

and modus ponens we get

$$\Gamma \vdash (\varphi \land \chi) \rightarrow \theta(a).$$

Since the eigenvariable condition still applies, we can add a step to this derivation justified by QR, and get

$$\Gamma \vdash (\varphi \land \chi) \rightarrow \forall x\, \theta(x).$$

We also have

$$\vdash ((\varphi \land \chi) \rightarrow \forall x\, \theta(x)) \rightarrow (\varphi \rightarrow (\chi \rightarrow \forall x\, \theta(x)),$$

so by modus ponens,

$$\Gamma \vdash \varphi \rightarrow (\chi \rightarrow \forall x\, \theta(x)),$$

i.e., $\Gamma \vdash \psi$.

We leave the case where ψ is justified by the rule QR, but is of the form $\exists x\, \theta(x) \rightarrow \chi$, as an exercise. □

Problem axd.1. Complete the proof of Theorem axd.1.

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Bibliography