int.1 The Strict Conditional

cnt:int:str:sec Lewis introduced the *strict conditional* $\strictif$ and argued that it, not the material conditional, corresponds to implication. In alethic modal logic, $\varphi \strictif \psi$ can be defined as $\Box(\varphi \to \psi)$. A strict conditional is thus true (at a world) iff the corresponding material conditional is necessary.

How does the strict conditional fare vis-a-vis the paradoxes of the material conditional? A strict conditional with a false antecedent and one with a true consequent, may be true, or it may be false. Moreover, $(\varphi \strictif \psi) \lor (\psi \strictif \varphi)$ is not valid. The strict conditional $\varphi \strictif \psi$ is also not equivalent to $\lnot\varphi \lor \psi$, so it is not truth functional.

We have:

$$\varphi \strictif \psi \vDash \lnot\varphi \lor \psi \text{ but:} \tag{1}$$
$$\lnot\varphi \lor \psi \nvDash \varphi \strictif \psi \tag{2}$$
$$\psi \nvDash \varphi \strictif \psi \tag{3}$$
$$\lnot\varphi \nvDash \varphi \strictif \psi \tag{4}$$
$$\lnot(\varphi \to \psi) \nvDash \varphi \land \lnot\psi \text{ but:} \tag{5}$$
$$\varphi \land \lnot\psi \vDash \lnot(\varphi \strictif \psi) \tag{6}$$

However, the strict conditional still supports modus ponens:

$$\varphi, \varphi \strictif \psi \vDash \psi \tag{7}$$

The strict conditional agglomerates:

$$\varphi \strictif \psi, \varphi \strictif \chi \vDash \varphi \strictif (\psi \land \chi) \tag{8}$$

Antecedent strengthening holds for the strict conditional:

$$\varphi \strictif \psi \vDash (\varphi \land \chi) \strictif \psi \tag{9}$$

The strict conditional is also transitive:

$$\varphi \strictif \psi, \psi \strictif \chi \vDash \varphi \strictif \chi \tag{10}$$

Finally, the strict conditional is equivalent to its contrapositive:

$$\varphi \strictif \psi \vDash \lnot\psi \strictif \lnot\varphi \tag{11}$$
$$\lnot\psi \strictif \lnot\varphi \vDash \varphi \strictif \psi \tag{12}$$

Problem int.1. Give **S5**-counterexamples to the entailment relations which do not hold for the strict conditional, i.e., for:

1. $\lnot p \nvDash \Box(p \to q)$

2. $q \nvDash \Box(p \to q)$

3. $\neg\Box(p \to q) \nvDash p \land \neg q$

4. $\nvDash \Box(p \to q) \lor \Box(q \to p)$

Problem int.2. Show that the valid entailment relations hold for the strict conditional by giving **S5**-proofs of:

1. $\Box(\varphi \to \psi) \vDash \neg\varphi \lor \psi$

2. $\varphi \land \neg\psi \vDash \neg\Box(\varphi \to \psi)$

3. $\varphi, \Box(\varphi \to \psi) \vDash \psi$

4. $\Box(\varphi \to \psi), \Box(\varphi \to \chi) \vDash \Box(\varphi \to (\psi \land \chi))$

5. $\Box(\varphi \to \psi) \vDash \Box((\varphi \land \chi) \to \psi)$

6. $\Box(\varphi \to \psi), \Box(\psi \to \chi) \vDash \Box(\varphi \to \chi)$

7. $\Box(\varphi \to \psi) \vDash \Box(\neg\psi \to \neg\varphi)$

8. $\Box(\neg\psi \to \neg\varphi) \vDash \Box(\varphi \to \psi)$

However, the strict conditional still has its own "paradoxes." Just as a material conditional with a false antecedent or a true consequent is true, a strict conditional with a *necessarily* false antecedent or a necessarily true consequent is true. Moreover, any true strict conditional is necessarily true, and any false strict conditional is necessarily false. In other words, we have

$$\Box\neg\varphi \vDash \varphi \prec \psi \tag{13}$$

$$\Box\psi \vDash \varphi \prec \psi \tag{14}$$

$$\varphi \prec \psi \vDash \Box(\varphi \prec \psi) \tag{15}$$

$$\neg(\varphi \prec \psi) \vDash \Box\neg(\varphi \prec \psi) \tag{16}$$

These are not problems if you think of $\prec$ as "implies." Logical entailment relationships are, after all, mathematical facts and so can't be contingent. But they do raise issues if you want to use $\prec$ as a logical connective that is supposed to capture "if ... then ...," especially the last two. For surely there are "if ... then ..." statements that are contingently true or contingently false—in fact, they generally are neither necessary nor impossible.

Problem int.3. Give proofs in **S5** of:

1. $\Box\neg\psi \vDash \varphi \prec \psi$

2. $\varphi \prec \psi \vDash \Box(\varphi \prec \psi)$

3. $\neg(\varphi \prec \psi) \vDash \Box\neg(\varphi \prec \psi)$

Use the definition of $\strictif$ to do so.

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Bibliography