

rec.1 Sequences

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sec

The set of primitive recursive functions is remarkably robust. But we will be able to do even more once we have developed a adequate means of handling *sequences*. We will identify finite sequences of natural numbers with natural numbers in the following way: the sequence $\langle a_0, a_1, a_2, \dots, a_k \rangle$ corresponds to the number

$$p_0^{a_0+1} \cdot p_1^{a_1+1} \cdot p_2^{a_2+1} \cdot \dots \cdot p_k^{a_k+1}.$$

We add one to the exponents to guarantee that, for example, the sequences $\langle 2, 7, 3 \rangle$ and $\langle 2, 7, 3, 0, 0 \rangle$ have distinct numeric codes. We can take both 0 and 1 to code the empty sequence; for concreteness, let Λ denote 0.

The reason that this coding of sequences works is the so-called Fundamental Theorem of Arithmetic: every natural number $n \geq 2$ can be written in one and only one way in the form

$$n = p_0^{a_0} \cdot p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$$

with $a_k \geq 1$. This guarantees that the mapping $\langle \rangle(a_0, \dots, a_k) = \langle a_0, \dots, a_k \rangle$ is injective: different sequences are mapped to different numbers; to each number only at most one sequence corresponds.

We'll now show that the operations of determining the length of a sequence, determining its i th element, appending an element to a sequence, and concatenating two sequences, are all primitive recursive.

Proposition rec.1. *The function $\text{len}(s)$, which returns the length of the sequence s , is primitive recursive.*

Proof. Let $R(i, s)$ be the relation defined by

$$R(i, s) \text{ iff } p_i \mid s \wedge p_{i+1} \nmid s.$$

R is clearly primitive recursive. Whenever s is the code of a non-empty sequence, i.e.,

$$s = p_0^{a_0+1} \cdot \dots \cdot p_k^{a_k+1},$$

$R(i, s)$ holds if p_i is the largest prime such that $p_i \mid s$, i.e., $i = k$. The length of s thus is $i + 1$ iff p_i is the largest prime that divides s , so we can let

$$\text{len}(s) = \begin{cases} 0 & \text{if } s = 0 \text{ or } s = 1 \\ 1 + (\min i < s) R(i, s) & \text{otherwise} \end{cases}$$

We can use bounded minimization, since there is only one i that satisfies $R(s, i)$ when s is a code of a sequence, and if i exists it is less than s itself. \square

Proposition rec.2. *The function $\text{append}(s, a)$, which returns the result of appending a to the sequence s , is primitive recursive.*

Proof. append can be defined by:

$$\text{append}(s, a) = \begin{cases} 2^{a+1} & \text{if } s = 0 \text{ or } s = 1 \\ s \cdot p_{\text{len}(s)}^{a+1} & \text{otherwise.} \end{cases} \quad \square$$

Proposition rec.3. *The function element(s, i), which returns the i th element of s (where the initial element is called the 0th), or 0 if i is greater than or equal to the length of s , is primitive recursive.*

Proof. Note that a is the i th element of s iff p_i^{a+1} is the largest power of p_i that divides s , i.e., $p_i^{a+1} \mid s$ but $p_i^{a+2} \nmid s$. So:

$$\text{element}(s, i) = \begin{cases} 0 & \text{if } i \geq \text{len}(s) \\ (\min a < s) (p_i^{a+2} \nmid s) & \text{otherwise.} \end{cases} \quad \square$$

Instead of using the official names for the functions defined above, we introduce a more compact notation. We will use $(s)_i$ instead of $\text{element}(s, i)$, and $\langle s_0, \dots, s_k \rangle$ to abbreviate

$$\text{append}(\text{append}(\dots \text{append}(A, s_0) \dots), s_k).$$

Note that if s has length k , the elements of s are $(s)_0, \dots, (s)_{k-1}$.

Proposition rec.4. *The function concat(s, t), which concatenates two sequences, is primitive recursive.*

Proof. We want a function concat with the property that

$$\text{concat}(\langle a_0, \dots, a_k \rangle, \langle b_0, \dots, b_l \rangle) = \langle a_0, \dots, a_k, b_0, \dots, b_l \rangle.$$

We'll use a "helper" function hconcat(s, t, n) which concatenates the first n symbols of t to s . This function can be defined by primitive recursion as follows:

$$\begin{aligned} \text{hconcat}(s, t, 0) &= s \\ \text{hconcat}(s, t, n+1) &= \text{append}(\text{hconcat}(s, t, n), (t)_n) \end{aligned}$$

Then we can define concat by

$$\text{concat}(s, t) = \text{hconcat}(s, t, \text{len}(t)). \quad \square$$

We will write $s \frown t$ instead of $\text{concat}(s, t)$.

It will be useful for us to be able to bound the numeric code of a sequence in terms of its length and its largest element. Suppose s is a sequence of length k , each element of which is less than or equal to some number x . Then s has at

most k prime factors, each at most p_{k-1} , and each raised to at most $x + 1$ in the prime factorization of s . In other words, if we define

$$\text{sequenceBound}(x, k) = p_{k-1}^{k \cdot (x+1)},$$

then the numeric code of the sequence s described above is at most $\text{sequenceBound}(x, k)$.

Having such a bound on sequences gives us a way of defining new functions using bounded search. For example, we can define `concat` using bounded search. All we need to do is write down a primitive recursive *specification* of the object (number of the concatenated sequence) we are looking for, and a bound on how far to look. The following works:

$$\begin{aligned} \text{concat}(s, t) = & (\min v < \text{sequenceBound}(s + t, \text{len}(s) + \text{len}(t))) \\ & (\text{len}(v) = \text{len}(s) + \text{len}(t) \wedge \\ & (\forall i < \text{len}(s)) ((v)_i = (s)_i) \wedge \\ & (\forall j < \text{len}(t)) ((v)_{\text{len}(s)+j} = (t)_j)) \end{aligned}$$

Problem rec.1. Show that there is a primitive recursive function `sconcat`(s) with the property that

$$\text{sconcat}(\langle s_0, \dots, s_k \rangle) = s_0 \frown \dots \frown s_k.$$

Problem rec.2. Show that there is a primitive recursive function `tail`(s) with the property that

$$\begin{aligned} \text{tail}(A) &= 0 \text{ and} \\ \text{tail}(\langle s_0, \dots, s_k \rangle) &= \langle s_1, \dots, s_k \rangle. \end{aligned}$$

cmp:rec:seq; **Proposition rec.5.** *The function `subseq`(s, i, n) which returns the subsequence of s of length n beginning at the i th element, is primitive recursive.*
prop:subseq

Proof. Exercise. □

Problem rec.3. Prove [Proposition rec.5](#).

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Bibliography