Suppose a function \( h \) is defined by primitive recursion
\[
\begin{align*}
 h(\vec{x}, 0) &= f(\vec{x}) \\
 h(\vec{x}, y+1) &= g(\vec{x}, y, h(\vec{x}, y))
\end{align*}
\]
and suppose the functions \( f \) and \( g \) are computable. (We use \( \vec{x} \) to abbreviate \( x_0, \ldots, x_{k-1} \).) Then \( h(\vec{x}, 0) \) can obviously be computed, since it is just \( f(\vec{x}) \) which we assume is computable. \( h(\vec{x}, 1) \) can then also be computed, since \( 1 = 0 + 1 \) and so \( h(\vec{x}, 1) \) is just
\[
h(\vec{x}, 1) = g(\vec{x}, 0, h(\vec{x}, 0)) = g(\vec{x}, 0, f(\vec{x})).
\]
We can go on in this way and compute
\[
\begin{align*}
 h(\vec{x}, 2) &= g(\vec{x}, 1, h(\vec{x}, 1)) = g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x}))) \\
 h(\vec{x}, 3) &= g(\vec{x}, 2, h(\vec{x}, 2)) = g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))) \\
 h(\vec{x}, 4) &= g(\vec{x}, 3, h(\vec{x}, 3)) = g(\vec{x}, 3, g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))))) \\
 &\vdots
\end{align*}
\]
Thus, to compute \( h(\vec{x}, y) \) in general, successively compute \( h(\vec{x}, 0), h(\vec{x}, 1), \ldots, \) until we reach \( h(\vec{x}, y) \).

Thus, a primitive recursive definition yields a new computable function if the functions \( f \) and \( g \) are computable. Composition of functions also results in a computable function if the functions \( f \) and \( g_i \) are computable.

Since the basic functions zero, succ, and \( P^n_i \) are computable, and composition and primitive recursion yield computable functions from computable functions, this means that every primitive recursive function is computable.

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Bibliography