

rec.1 Primitive Recursive Functions are Computable

`cmp:rec:cmp:` Suppose a function h is defined by primitive recursion
`sec`

$$\begin{aligned}h(\vec{x}, 0) &= f(\vec{x}) \\h(\vec{x}, y + 1) &= g(\vec{x}, y, h(\vec{x}, y))\end{aligned}$$

and suppose the functions f and g are computable. (We use \vec{x} to abbreviate x_0, \dots, x_{k-1} .) Then $h(\vec{x}, 0)$ can obviously be computed, since it is just $f(\vec{x})$ which we assume is computable. $h(\vec{x}, 1)$ can then also be computed, since $1 = 0 + 1$ and so $h(\vec{x}, 1)$ is just

$$h(\vec{x}, 1) = g(\vec{x}, 0, h(\vec{x}, 0)) = g(\vec{x}, 0, f(\vec{x})).$$

We can go on in this way and compute

$$\begin{aligned}h(\vec{x}, 2) &= g(\vec{x}, 1, h(\vec{x}, 1)) = g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x}))) \\h(\vec{x}, 3) &= g(\vec{x}, 2, h(\vec{x}, 2)) = g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))) \\h(\vec{x}, 4) &= g(\vec{x}, 3, h(\vec{x}, 3)) = g(\vec{x}, 3, g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))))) \\&\vdots\end{aligned}$$

Thus, to compute $h(\vec{x}, y)$ in general, successively compute $h(\vec{x}, 0), h(\vec{x}, 1), \dots$, until we reach $h(\vec{x}, y)$.

Thus, a primitive recursive definition yields a new computable function if the functions f and g are computable. Composition of functions also results in a computable function if the functions f and g_i are computable.

Since the basic functions zero, succ, and P_i^n are computable, and composition and primitive recursion yield computable functions from computable functions, this means that every primitive recursive function is computable.

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Bibliography