

## rec.1 Primitive Recursive Functions are Computable

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sec Suppose a function  $h$  is defined by primitive recursion

$$\begin{aligned}h(\vec{x}, 0) &= f(\vec{x}) \\h(\vec{x}, y) &= g(\vec{x}, y, h(\vec{x}, y))\end{aligned}$$

and suppose the functions  $f$  and  $g$  are computable. (We use  $\vec{x}$  to abbreviate  $x_0, \dots, x_{k-1}$ .) Then  $h(\vec{x}, 0)$  can obviously be computed, since it is just  $f(\vec{x})$  which we assume is computable.  $h(\vec{x}, 1)$  can then also be computed, since  $1 = 0 + 1$  and so  $h(\vec{x}, 1)$  is just

$$h(\vec{x}, 1) = g(\vec{x}, 0, h(\vec{x}, 0)) = g(\vec{x}, 0, f(\vec{x})).$$

We can go on in this way and compute

$$\begin{aligned}h(\vec{x}, 2) &= g(\vec{x}, 1, h(\vec{x}, 1)) = g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x}))) \\h(\vec{x}, 3) &= g(\vec{x}, 2, h(\vec{x}, 2)) = g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))) \\h(\vec{x}, 4) &= g(\vec{x}, 3, h(\vec{x}, 3)) = g(\vec{x}, 3, g(\vec{x}, 2, g(\vec{x}, 1, g(\vec{x}, 0, f(\vec{x})))))) \\&\vdots\end{aligned}$$

Thus, to compute  $h(\vec{x}, y)$  in general, successively compute  $h(\vec{x}, 0), h(\vec{x}, 1), \dots$ , until we reach  $h(\vec{x}, y)$ .

Thus, a primitive recursive definition yields a new computable function if the functions  $f$  and  $g$  are computable. Composition of functions also results in a computable function if the functions  $f$  and  $g_i$  are computable.

Since the basic functions zero, succ, and  $P_i^n$  are computable, and composition and primitive recursion yield computable functions from computable functions, this means that every primitive recursive function is computable.

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## Bibliography