One advantage to having the precise inductive description of the primitive recursive functions is that we can be systematic in describing them. For example, we can assign a “notation” to each such function, as follows. Use symbols $\text{zero}$, $\text{succ}$, and $P^n_i$ for zero, successor, and the projections. Now suppose $h$ is defined by composition from a $k$-place function $f$ and $n$-place functions $g_0, \ldots, g_{k-1}$, and we have assigned notations $F, G_0, \ldots, G_{k-1}$ to the latter functions. Then, using a new symbol $\text{Comp}_{k,n}$, we can denote the function $h$ by $\text{Comp}_{k,n}[F, G_0, \ldots, G_{k-1}]$.

For functions defined by primitive recursion, we can use analogous notations. Suppose the $(k+1)$-ary function $h$ is defined by primitive recursion from the $k$-ary function $f$ and the $(k+2)$-ary function $g$, and the notations assigned to $f$ and $g$ are $F$ and $G$, respectively. Then the notation assigned to $h$ is $\text{Rec}_{k}[F, G]$.

Recall that the addition function is defined by primitive recursion as

\[
\begin{align*}
\text{add}(x_0, 0) &= P^1_0(x_0) = x_0 \\
\text{add}(x_0, y + 1) &= \text{succ}(P^3_2(x_0, y, \text{add}(x_0, y))) \equiv \text{add}(x_0, y) + 1
\end{align*}
\]

Here the role of $f$ is played by $P^1_0$, and the role of $g$ is played by $\text{succ}(P^3_2(x_0, y, z))$, which is assigned the notation $\text{Comp}_{1,3}[\text{succ}, P^3_2]$ as it is the result of defining a function by composition from the 1-ary function $\text{succ}$ and the 3-ary function $P^3_2$.

With this setup, we can denote the addition function by

$$\text{Rec}_{1}[P_0^1, \text{Comp}_{1,3}[\text{succ}, P_2^3]].$$

Having these notations sometimes proves useful, e.g., when enumerating primitive recursive functions.

**Problem rec.1.** Give the complete primitive recursive notation for $\text{mult}$.

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**Bibliography**