rec.1 Examples of Primitive Recursive Functions

cmp:rec:exa:sec We already have some examples of primitive recursive functions: the addition and multiplication functions add and mult. The identity function $\mathrm{id}(x) = x$ is primitive recursive, since it is just P^1_0. The constant functions $\mathrm{const}_n(x) = n$ are primitive recursive since they can be defined from zero and succ by successive composition. This is useful when we want to use constants in primitive recursive definitions, e.g., if we want to define the function $f(x) = 2 \cdot x$ can obtain it by composition from $\mathrm{const}_n(x)$ and multiplication as $f(x) = \mathrm{mult}(\mathrm{const}_2(x), P^1_0(x))$. We'll make use of this trick from now on.

Proposition rec.1. *The exponentiation function* $\exp(x, y) = x^y$ *is primitive recursive.*

Proof. We can define exp primitive recursively as

$$\begin{aligned}
\exp(x, 0) &= 1 \\
\exp(x, y + 1) &= \mathrm{mult}(x, \exp(x, y)).
\end{aligned}$$

Strictly speaking, this is not a recursive definition from primitive recursive functions. Officially, though, we have:

$$\begin{aligned}
\exp(x, 0) &= f(x) \\
\exp(x, y + 1) &= g(x, y, \exp(x, y)).
\end{aligned}$$

where

$$\begin{aligned}
f(x) &= \mathrm{succ}(\mathrm{zero}(x)) = 1 \\
g(x, y, z) &= \mathrm{mult}(P^3_0(x, y, z), P^3_2(x, y, z) = x \cdot z
\end{aligned}$$

and so f and g are defined from primitive recursive functions by composition. □

Proposition rec.2. *The predecessor function* $\mathrm{pred}(y)$ *defined by*

$$\mathrm{pred}(y) = \begin{cases} 0 & \text{if } y = 0 \\ y - 1 & \text{otherwise} \end{cases}$$

is primitive recursive.

Proof. Note that

$$\begin{aligned}
\mathrm{pred}(0) &= 0 \text{ and} \\
\mathrm{pred}(y + 1) &= y.
\end{aligned}$$

This is almost a primitive recursive definition. It does not, strictly speaking, fit into the pattern of definition by primitive recursion, since that pattern requires at least one extra argument x. It is also odd in that it does not actually use $\mathrm{pred}(y)$ in the definition of $\mathrm{pred}(y+1)$. But we can first define $\mathrm{pred}'(x,y)$ by

$$\begin{aligned}
\mathrm{pred}'(x,0) &= \mathrm{zero}(x) = 0, \\
\mathrm{pred}'(x,y+1) &= P^3_1(x,y,\mathrm{pred}'(x,y)) = y.
\end{aligned}$$

and then define pred from it by composition, e.g., as $\mathrm{pred}(x) = \mathrm{pred}'(\mathrm{zero}(x), P^1_0(x))$. □

Proposition rec.3. *The factorial function* $\mathrm{fac}(x) = x\,! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot x$ *is primitive recursive.*

Proof. The obvious primitive recursive definition is

$$\begin{aligned}
\mathrm{fac}(0) &= 1 \\
\mathrm{fac}(y+1) &= \mathrm{fac}(y) \cdot (y+1).
\end{aligned}$$

Officially, we have to first define a two-place function h

$$\begin{aligned}
h(x,0) &= \mathrm{const}_1(x) \\
h(x,y) &= g(x,y,h(x,y))
\end{aligned}$$

where $g(x,y,z) = \mathrm{mult}(P^3_2(x,y,z), \mathrm{succ}(P^3_1(x,y,z)$ and then let

$$\mathrm{fac}(y) = h(P^1_0(y), P^1_0(y))$$

From now on we'll be a bit more laissez-faire and not give the official definitions by composition and primitive recursion. □

Proposition rec.4. *Truncated subtraction,* $x \dot{-} y$*, defined by*

$$x \dot{-} y = \begin{cases} 0 & \text{if } x > y \\ x - y & \text{otherwise} \end{cases}$$

is primitive recursive.

Proof. We have:

$$\begin{aligned}
x \dot{-} 0 &= x \\
x \dot{-} (y+1) &= \mathrm{pred}(x \dot{-} y)
\end{aligned}$$

□

Proposition rec.5. *The distance between* x *and* y*,* $|x-y|$*, is primitive recursive.*

Proof. We have $|x-y| = (x \dot{-} y) + (y \dot{-} x)$, so the distance can be defined by composition from $+$ and $\dot{-}$, which are primitive recursive. □

Proposition rec.6. *The maximum of x and y, $\max(x, y)$, is primitive recursive.*

Proof. We can define $\max(x, y)$ by composition from $+$ and $\dot{-}$ by

$$\max(x, y) = x + (y \dot{-} x).$$

If x is the maximum, i.e., $x \geq y$, then $y \dot{-} x = 0$, so $x + (y \dot{-} x) = x + 0 = x$. If y is the maximum, then $y \dot{-} x = y - x$, and so $x + (y \dot{-} x) = x + (y - x) = y$. $\square$

cmp:rec:exa:prop:min-pr **Proposition rec.7.** *The minimum of x and y, $\min(x, y)$, is primitive recursive.*

Proof. Exercise. $\square$

Problem rec.1. Prove Proposition rec.7.

Problem rec.2. Show that

$$f(x, y) = \left. 2^{(2^{\cdot^{\cdot^{\cdot^{2^x}}}})} \right\} y \text{ 2's}$$

is primitive recursive.

Problem rec.3. Show that integer division $d(x, y) = \lfloor x/y \rfloor$ (i.e., division, where you disregard everything after the decimal point) is primitive recursive. When $y = 0$, we stipulate $d(x, y) = 0$. Give an explicit definition of d using primitive recursion and composition.

Proposition rec.8. *The set of primitive recursive functions is closed under the following two operations:*

1. *Finite sums: if $f(\vec{x}, z)$ is primitive recursive, then so is the function*

$$g(\vec{x}, y) = \sum_{z=0}^{y} f(\vec{x}, z).$$

2. *Finite products: if $f(\vec{x}, z)$ is primitive recursive, then so is the function*

$$h(\vec{x}, y) = \prod_{z=0}^{y} f(\vec{x}, z).$$

Proof. For example, finite sums are defined recursively by the equations

$$\begin{aligned} g(\vec{x}, 0) &= f(\vec{x}, 0) \\ g(\vec{x}, y + 1) &= g(\vec{x}, y) + f(\vec{x}, y + 1). \end{aligned}$$

$\square$

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Bibliography