

rec.1 Composition

cmp:rec:com:
sec If f and g are two one-place functions of natural numbers, we can compose them: $h(x) = g(f(x))$. The new function $h(x)$ is then defined by *composition* from the functions f and g . We'd like to generalize this to functions of more than one argument.

Here's one way of doing this: suppose f is a k -place function, and g_0, \dots, g_{k-1} are k functions which are all n -place. Then we can define a new n -place function h as follows:

$$h(x_0, \dots, x_{n-1}) = f(g_0(x_0, \dots, x_{n-1}), \dots, g_{k-1}(x_0, \dots, x_{n-1}))$$

If f and all g_i are computable, so is h : To compute $h(x_0, \dots, x_{n-1})$, first compute the values $y_i = g_i(x_0, \dots, x_{n-1})$ for each $i = 0, \dots, k-1$. Then feed these values into f to compute $h(x_0, \dots, x_{n-1}) = f(y_0, \dots, y_{k-1})$.

This may seem like an overly restrictive characterization of what happens when we compute a new function using some existing ones. For one thing, sometimes we do not use all the arguments of a function, as when we defined $g(x, y, z) = \text{succ}(z)$ for use in the primitive recursive definition of add. Suppose we are allowed use of the following functions:

$$P_i^n(x_0, \dots, x_{n-1}) = x_i$$

The functions P_i^k are called *projection* functions: P_i^n is an n -place function. Then g can be defined by

$$g(x, y, z) = \text{succ}(P_2^3).$$

Here the role of f is played by the 1-place function succ , so $k = 1$. And we have one 3-place function P_2^3 which plays the role of g_0 . The result is a 3-place function that returns the successor of the third argument.

The projection functions also allow us to define new functions by reordering or identifying arguments. For instance, the function $h(x) = \text{add}(x, x)$ can be defined by

$$h(x_0) = \text{add}(P_0^1(x_0), P_0^1(x_0)).$$

Here $k = 2$, $n = 1$, the role of $f(y_0, y_1)$ is played by add , and the roles of $g_0(x_0)$ and $g_1(x_0)$ are both played by $P_0^1(x_0)$, the one-place projection function (aka the identity function).

If $f(y_0, y_1)$ is a function we already have, we can define the function $h(x_0, x_1) = f(x_1, x_0)$ by

$$h(x_0, x_1) = f(P_1^2(x_0, x_1), P_0^2(x_0, x_1)).$$

Here $k = 2$, $n = 2$, and the roles of g_0 and g_1 are played by P_1^2 and P_0^2 , respectively.

You may also worry that g_0, \dots, g_{k-1} are all required to have the same arity n . (Remember that the *arity* of a function is the number of arguments; an n -place function has arity n .) But adding the projection functions provides

the desired flexibility. For example, suppose f and g are 3-place functions and h is the 2-place function defined by

$$h(x, y) = f(x, g(x, x, y), y).$$

The definition of h can be rewritten with the projection functions, as

$$h(x, y) = f(P_0^2(x, y), g(P_0^2(x, y), P_0^2(x, y), P_1^2(x, y)), P_1^2(x, y)).$$

Then h is the composition of f with P_0^2 , l , and P_1^2 , where

$$l(x, y) = g(P_0^2(x, y), P_0^2(x, y), P_1^2(x, y)),$$

i.e., l is the composition of g with P_0^2 , P_0^2 , and P_1^2 .

Photo Credits

Bibliography