If $f$ and $g$ are two one-place functions of natural numbers, we can compose them: $h(x) = g(f(x))$. The new function $h(x)$ is then defined by composition from the functions $f$ and $g$. We’d like to generalize this to functions of more than one argument.

Here’s one way of doing this: suppose $f$ is a $k$-place function, and $g_0, \ldots, g_{k-1}$ are $k$ functions which are all $n$-place. Then we can define a new $n$-place function $h$ as follows:

$$h(x_0, \ldots, x_{n-1}) = f(g_0(x_0, \ldots, x_{n-1}), \ldots, g_{k-1}(x_0, \ldots, x_{n-1}))$$

If $f$ and all $g_i$ are computable, so is $h$: To compute $h(x_0, \ldots, x_{n-1})$, first compute the values $y_i = g_i(x_0, \ldots, x_{n-1})$ for each $i = 0, \ldots, k - 1$. Then feed these values into $f$ to compute $h(x_0, \ldots, x_{k-1}) = f(y_0, \ldots, y_{k-1})$.

This may seem like an overly restrictive characterization of what happens when we compute a new function using some existing ones. For one thing, sometimes we do not use all the arguments of a function, as when we defined $g(x, y, z) = \text{succ}(z)$ for use in the primitive recursive definition of add. Suppose we are allowed use of the following functions:

$$P^k_n(x_0, \ldots, x_{n-1}) = x_i$$

The functions $P^k_n$ are called projection functions: $P^n_i$ is an $n$-place function. Then $g$ can be defined by

$$g(x, y, z) = \text{succ}(P^3_2(x, y, z)).$$

Here the role of $f$ is played by the 1-place function succ, so $k = 1$. And we have one 3-place function $P^3_2$ which plays the role of $g_0$. The result is a 3-place function that returns the successor of the third argument.

The projection functions also allow us to define new functions by reordering or identifying arguments. For instance, the function $h(x) = \text{add}(x, x)$ can be defined by

$$h(x_0) = \text{add}(P^1_0(x_0), P^1_0(x_0)).$$

Here $k = 2$, $n = 1$, the role of $f(y_0, y_1)$ is played by add, and the roles of $g_0(x_0)$ and $g_1(x_0)$ are both played by $P^1_0(x_0)$, the one-place projection function (aka the identity function).

If $f(y_0, y_1)$ is a function we already have, we can define the function $h(x_0, x_1) = f(x_1, x_0)$ by

$$h(x_0, x_1) = f(P^2_1(x_0, x_1), P^2_0(x_0, x_1)).$$

Here $k = 2$, $n = 2$, and the roles of $g_0$ and $g_1$ are played by $P^2_2$ and $P^2_0$, respectively.

You may also worry that $g_0, \ldots, g_{k-1}$ are all required to have the same arity $n$. (Remember that the arity of a function is the number of arguments; an $n$-place function has arity $n$.) But adding the projection functions provides...
the desired flexibility. For example, suppose $f$ and $g$ are 3-place functions and $h$ is the 2-place function defined by

$$h(x, y) = f(x, g(x, x, y), y).$$

The definition of $h$ can be rewritten with the projection functions, as

$$h(x, y) = f(P^2_0(x, y), g(P^2_0(x, y), P^2_0(x, y), P^2_1(x, y)), P^2_1(x, y)).$$

Then $h$ is the composition of $f$ with $P^2_0$, $l$, and $P^2_1$, where

$$l(x, y) = g(P^2_0(x, y), P^2_0(x, y), P^2_1(x, y)),$$

i.e., $l$ is the composition of $g$ with $P^2_0$, $P^2_0$, and $P^2_1$.

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**Bibliography**